Applied Operations Research

Simplex method at a glance

- 2018/2019 -

1. A farmer owns 35 acres of land and is going to plant wheat or corn. Each acre planted with wheat yields \$200 profit; each acre planted with corn yields \$300 profit. The labor and fertilizer used for each acre are given in the table below:

	Wheat	Corn
Labor	3 workers	2 workers
Fertilizer	2 tons	4 tons

One hundred workers and 120 tons of fertilizer are available. The problem is to determine how the farmer can maximize the profit from his land.

(a) Formulate the problem in linear programming.

(b) Graphically represent the feasible region.

(c) Determine an optimal solution and the corresponding optimal value.

(d) Use the Excel Solver to solve the problem.

(e) The Excel Solver provides a Sensitivity Report. For the objective function coefficients it gives the following values:

Variable Cells

...	Objective Coefficient	Allowable Increase	Allowable Decrease
...	200	100	50
...	300	100	100

For each objective function coefficient, show how to obtain the values in columns Allowable Increase and Allowable Decrease.

(f) Fix the wheat profit to \$200. For which value(s) of corn profit does the LP problem have optimal solutions alternative to the one found in b)?

2. Consider the following linear programming model

$$\begin{aligned}
\text{Max} \quad & Z = x_1 - x_2 + x_3 \\
\text{s.t.} \quad & 2x_1 - x_2 + 2x_3 \leq 6 \\
& -2x_1 + 4x_2 - x_3 \geq \alpha \qquad (\alpha \in \mathbb{R}) \\
& x_1 - x_2 + 2x_3 \geq 4 \\
& x_1, \quad x_2, \quad x_3 \geq 0
\end{aligned}$$

and the point $P = (0, 2, 4)$.

(a) Write the problem in the standard form.

(b) Find, if any, a value for α such that P is the vertex of the feasible region and give the corresponding value of the objective function.

(c) Consider that P is an optimal solution of the problem for the α value found previously. Comment the following sentence: "The plan $x_1 - x_2 + x_3 = 3$ intercepts the feasible region of the problem."

3. Consider the following linear programming model

$$\begin{array}{llcl} \text{Max} & Z = x_1 + 2x_2 & - & x_3 \\ \text{s.t.} & 2x_1 + 4x_2 & + 3x_3 & \geq 8 \\ & x_1 + x_2 & & \leq 6 \\ & -x_1 + x_2 & & \leq 4 \\ & x_1 & + \; x_3 & \leq 4 \\ & x_1, \quad x_2, & x_3 & \geq 0 \end{array}$$

(a) Write the problem in the standard form.

(b) Find an optimal solution of the problem that is obtained from the initial problem by adding the restriction $x_3 = 0$ and indicate the corresponding binding constraints.

(c) Does the solution of the previous question correspond to a vertex of the feasible region of the initial problem?

4. A farmer has the following resource endowments: 500 acres of land, 1500 hours of family labour and 120000€ of capital investment. She can use these resources to grow the following crops: corn, sorghum, wheat and soybeans. Any crop should not occupy more than 250 acres. Assume that the farmer works to maximize profit (revenue minus costs) from the production of these crops. She expects the following in terms of prices, crop yields, costs and labor requirements.

(a) Formulate the problem as a linear program.

Crop	Price (€/bushel)	Yield (bushel/acre)	Cost (€/acre)	Labor Requirement (hours/acre)
Corn	2.75	120	250	3.25
Sorghum	2.65	100	200	3.00
Wheat	3.15	105	245	3.15
Soybeans	6.75	45	230	3.30

(b) Solve the problem using the Excel Solver. What is the optimal solution to the problem?

(c) Indicate a basic feasible solution of the problem in the standard form.

5. Consider the problem P in linear programming associated to the following standard formulation:

Max $Z = 2x_1 + 3x_2$ given the feasible region

$$\mathcal{F} = \{(x_1, x_2, x_3, x_4, x_5) : \left| \begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\ x_1 & - & x_2 & & & - & x_4 & & & = & -2 \\ -2x_1 & + & x_2 & & & & & - & x_5 & = & -10 \\ x_1, & & x_2, & & x_3 & & x_4, & & x_5 & \geq & 0 \end{array} \right. \}$$

where x_1 and x_2 are the decision variables of P.

(a) Write P.

(b) Graph the feasible region of P.

(c) Find an optimal solution of P and indicate the corresponding basic feasible solution.

(d) Consider set $S = \{(x_1, x_2, x_3, x_4, x_5) \in \mathcal{F} : x_3,\ x_4,\ x_5 > 0\}$. How many vertices of the feasible region of P correspond to elements of S?

(e) Consider that the Simplex method is applied to P, starting at a basic feasible solution from S. Indicate which vertices the algorithm passes through.

6. Consider the following problem in linear programming:

$$\text{Min} \quad Z = 2x_1 + x_2 \quad + 2x_3$$

$$
\begin{array}{lrcll}
\text{s.t.} & x_1 + x_2 & + & x_3 & \leq 4 \\
 & x_1 + x_2 & + & x_3 & \geq 1 \\
 & x_1 & & & \leq 2 \\
 & & & x_3 & \leq 3 \\
 & 3x_2 & + & x_3 & \leq 6 \\
 & x_1, \quad x_2, & & x_3 & \geq 0
\end{array}
$$

(a) Write the problem in the standard form.

(b) Show that (1,0,3) is a vertex of the feasible region of the problem.

(c) Consider that constraint $x_1+x_2+x_3 \leq 4$ is replaced by $x_1+x_2+x_3 = 4$. Will (1,0,3) be an optimal solution of the problem?

7. Consider the following problem in linear programming:

$$
\begin{array}{lrcrcrl}
\text{Max} \quad Z = & 2x_1 + x_2 & - & x_3 & + & 3x_4 & \\
\text{s.t.} & x_2 & - & 2x_3 & + & x_4 & \geq 3 \\
 & x_1 & - & 2x_3 & + & x_4 & \geq 2 \\
 & x_1 & + & x_3 & & & \leq 3 \\
 & x_1 + x_2 & - & 2x_3 & + & x_4 & = 5 \\
 & x_1, \quad x_2, & & x_3, & & x_4 & \geq 0
\end{array}
$$

(a) Write the problem in the standard form.

(b) Verify that $(2, 3, 0, 0)$ is a vertex of the feasible region of the problem and indicate the corresponding value of the objective function value.

8. Consider the following problem in linear programming:

$$
\begin{aligned}
\text{Max} \quad & Z = 4x_1 + x_2 + 2x_3 + 3x_4 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \leq \frac{3}{2} \\
& x_1 + x_2 \geq x_3 \\
& x_2 + 2x_4 = 2 \\
& x_1, \; x_2, \; x_3, \; x_4 \geq 0
\end{aligned}
$$

(a) Write the problem in the standard form.

(b) Verify that $(0,\ 1,\ 0,\ \frac{1}{2})$ is a vertex of the feasible region of the problem.

(c) Find an optimal solution of the problem that is obtained from the initial problem by adding the restriction $x_4 = 1$ and indicate the corresponding objective function value.

(d) Justify that solution $(0, 1, 0, \frac{1}{2})$ is not an optimal solution of the initial problem.

9. Consider the problem in linear programming

$$
\begin{aligned}
\text{Max} \quad & Z = 20x_1 + 30x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 120 \\
& x_1 \leq 60 \\
& x_2 \leq 50 \\
& x_1, \; x_2 \geq 0
\end{aligned}
$$

(a) Graph the feasible region and the feasible solutions with the objective function value of 600.

(b) Find an optimal solution and indicate the corresponding basic feasible solution. Which variables are basic in this basic feasible solution?

(c) Indicate two adjacent basic feasible solutions.

(d) What is the largest range of variation in the RHS of the third constraint that maintains optimal the solution referred to in question b)?

(e) What is the range of optimality for the coefficient of x_1 of the objective function?

(f) If the coefficients of x_1 and x_2 in the objective function were equal and positive, what would be the optimal solutions?

10. Consider the problem P in linear programming associated to the following standard formulation:

$$
\begin{array}{llcl}
\text{Max} & Z = 50000x_1 + 120000x_2 + 150000x_3 & & \\
\text{s.t.} & x_1 + x_2 + x_3 & = & 100 \\
 & x_1 - x_4 & = & 20 \\
 & x_2 - x_5 & = & 30 \\
 & x_3 - x_6 & = & 10 \\
 & 30000x_1 + 50000x_2 + 40000x_3 + x_7 & = & 65000000
\end{array}
$$

$$240x_1 + 480x_2 + 300x_3 + x_8 = 40000$$

$$x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6, \quad x_7, \quad x_8 \geq 0$$

where x_1, x_2 and x_3 are the decision variables of P. Let $u = (20, 30, 50, 0, 0, 40, 24000000, 5800)$ and $v = (60, 30, 10, 40, 0, 0, 28000000, 8200)$ be two basic feasible solutions.

(a) Indicate the solution set of the system

$$\begin{cases} x_1 + x_2 + x_3 = 100 \\ x_1 = 20 \\ x_2 = 30 \\ x_3 - x_6 = 10 \\ 30000x_1 + 50000x_2 + 40000x_3 + x_7 = 65000000 \\ 240x_1 + 480x_2 + 300x_3 + x_8 = 40000 \end{cases}.$$

(b) Do u and v correspond to adjacent vertices of the feasible region of P?

(c) Write P.

11. Consider the problem P in linear programming associated to the following standard formulation:

$$\text{Max} \quad Z = 29x_1 + 45x_2$$

$$\text{s.t.} \quad 2x_1 + 8x_2 + x_3 = 60$$

$$\text{(s)} \quad 4x_1 + 4x_2 + x_4 = 60$$

$$x_1, \quad x_2, \quad\quad x_3, \quad x_4 \geq 0$$

where x_1 and x_2 are the decision variables of P.

Consider that the Simplex algorithm is applied to P, starting at the basic feasible solution where x_3 and x_4 are the basic variables. Let $\begin{cases} x_3 &= 60 - 2x_1 - 8x_2 \\ x_4 &= 60 - 4x_1 - 4x_2 \end{cases}$ be the system (s) solved for these variables.

(a) Indicate the vertex where the Simplex algorithm starts.

(b) Find which vertex the algorithm goes next and indicate the corresponding objective function value.

12. Consider that the Simplex algorithm is applied to a maximization problem with two variables, x_1 and x_2, and two inequalities. Let x_3 and x_4 denote the slack variables associated to these constraints. Suppose that the current basic feasible solution has x_1 and x_3 as basic variables. Let $\begin{cases} x_1 &= 15 - x_2 - \frac{1}{4}x_4 \\ x_3 &= 30 - 6x_2 + \frac{1}{2}x_4 \end{cases}$ be the system of constraints in the standard form and $Z = 435 + 16x_2 - 7.25x_4$ the objective function, both expressed in terms of the non-basic variables.

(a) Indicate the current vertex.

(b) Find which vertex the algorithm goes next and indicate the corresponding objective function value.

13. Consider that the Simplex algorithm is applied to a maximization problem in linear programming with two variables, x_1 and x_2, and two inequalities. Let x_3 and x_4 denote the slack variables associated to these constraints. Suppose that the current basic feasible solution has x_1 and x_2 as basic variables. Let $\left\{ \begin{array}{lcl} x_1 & = & 10 + \frac{1}{6}x_3 - \frac{1}{3}x_4 \\ x_2 & = & 5 - \frac{1}{6}x_3 + \frac{1}{12}x_4 \end{array} \right.$ be the system of constraints in the standard form and $Z = 515 - 2.667x_3 - 5.917x_4$ the objective function, both expressed in terms of the non-basic variables.

(a) Indicate the current vertex.

(b) Is it possible that the algorithm goes to another vertex?