Growth Functions

Margarida Tomé, Susana Barreiro
Instituto Superior de Agronomia
Universidade de Lisboa

Outline

- Growth functions
- Theoretical growth functions
  - ➔ Lundqvist-Korf type functions
  - ➔ Richards type functions
  - ➔ Hossfeld IV function
    - McDill-Amateis function
- Zeide decomposition of growth functions
- Simultaneous modeling of several individuals (trees or stands)
- Formulating growth functions without age explicit

Growth functions

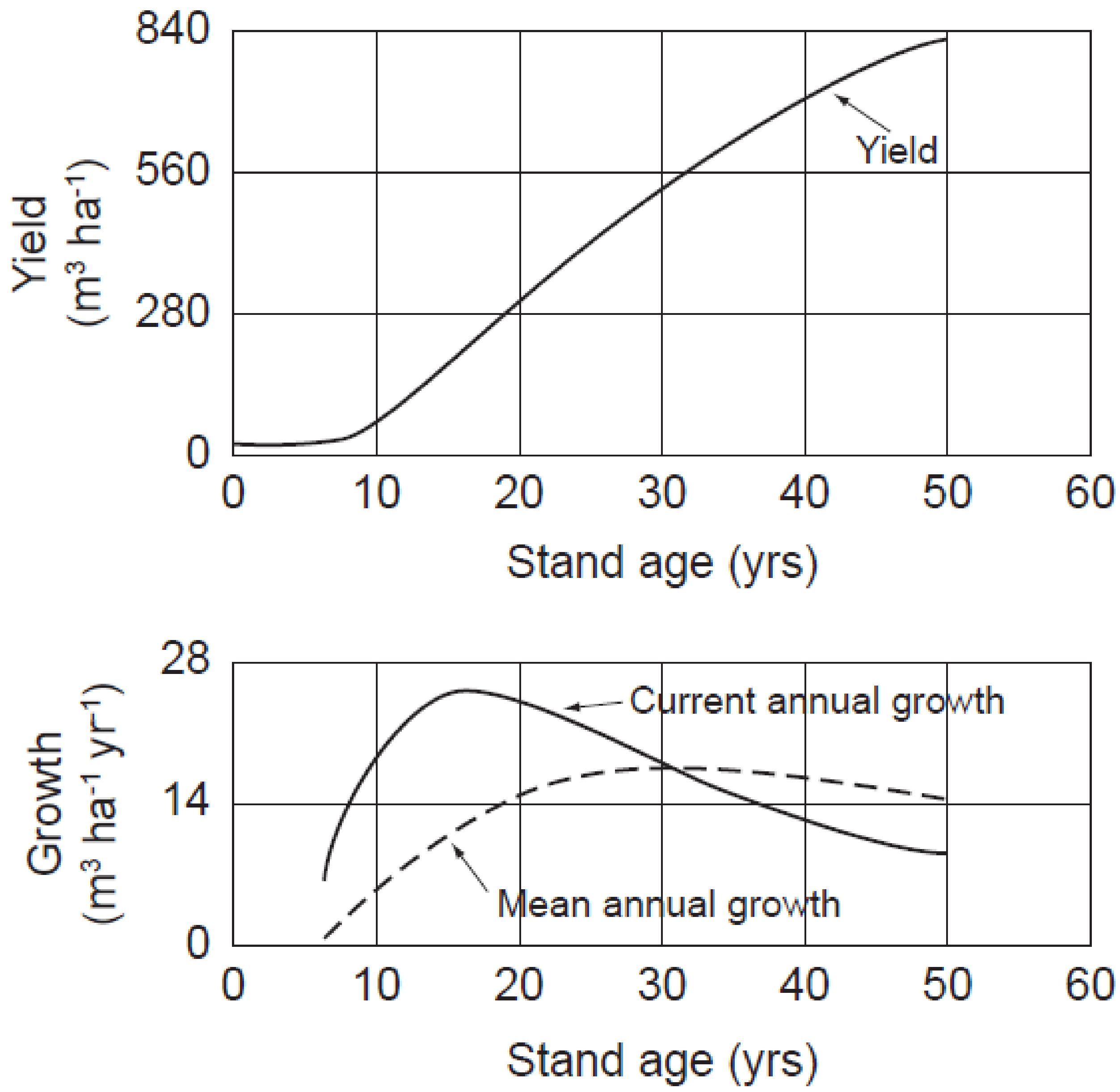
- **The selection of functions – *growth functions* - appropriate to model tree and stand growth is an essencial stage in the development of growth models**

→ Differencial form (growth)

$$\frac{dy}{dt} = f(t)$$

→ Integral form (yield)

$$y = \int f(t)\,dt$$


840
560
280
0
Yield
Yield
(m3 ha-1)
0
10
20
30
40
50
60
Stand age (yrs)
28
14
0
Current annual growth
Mean annual growth
Growth
(m3 ha-1 yr-1)
0
10
20
30
40
50
60
Stand age (yrs)

Growth functions

- **Growth functions must have a shape that is in accordance with the principles of biological growth:**

  - ➔ **The curve is limited by yield 0 at the start (t=0 ou t=t_0) and by a maximum yield at an advanced age (existence of assymptote)**

  - ➔ **the relative growth rate (variation of the x variable per unit of time and unit of x) presents a maximum at a very early stage, decreasing afterwards; in most cases, the maximum occurs very early so that we can use decreasing functions to model relative growth rate**

  - ➔ **The slope of the curve increases in the initial stage and decreases after a certain point in time (existence of an inflexion point)**

Growth functions

- **Two types of functions have been used to model growth:**

→ ***Empirical growth functions***

- Relationship between the dependent variable – the one we want to model – and the regressors according to some mathematical function – e.g. linear, parabolic, without trying to identify the causes or explaining the phenomenon

→ ***Functional or theoretical growth functions***

- Conceived in terms of the mechanism of forest growth, usually having an underlying hypothesis associated with the principles of forest growth

Theoretical growth functions

- Theoretical growth functions have commonly been developed in their growth form - either absolute or relative growth - and the respective yield form has been obtained by integration

- Generally this approach allows interpretation of the function parameters and helps to impose restrictions on the values that the parameters can take to be biologically consistent

- Theoretical growth functions are grouped according to their functional form in:

  - ➔ **Lundqvist-Korf type**
  - ➔ **Richards type**
  - ➔ **Hossfeld IV type**
  - ➔ Other growth functions

- **Theoretical growth functions**

Lundqvist-Korf type functions

- **Differential form:**

→ **Based on the hypothesis that the relative growth rate has a linear relationship with the inverse of time^{m+1} (which means that it decreases nonlinearly with time):**

$$\frac{1}{Y}\frac{dY}{dt} = k\frac{m}{t^{(m+1)}} \quad \Leftrightarrow \quad \frac{1}{Y}dY = -kd\frac{1}{t^{m}}$$

→ **Schumacher function if m=1**

Lundqvist-Korf type functions

- **Integral form:**

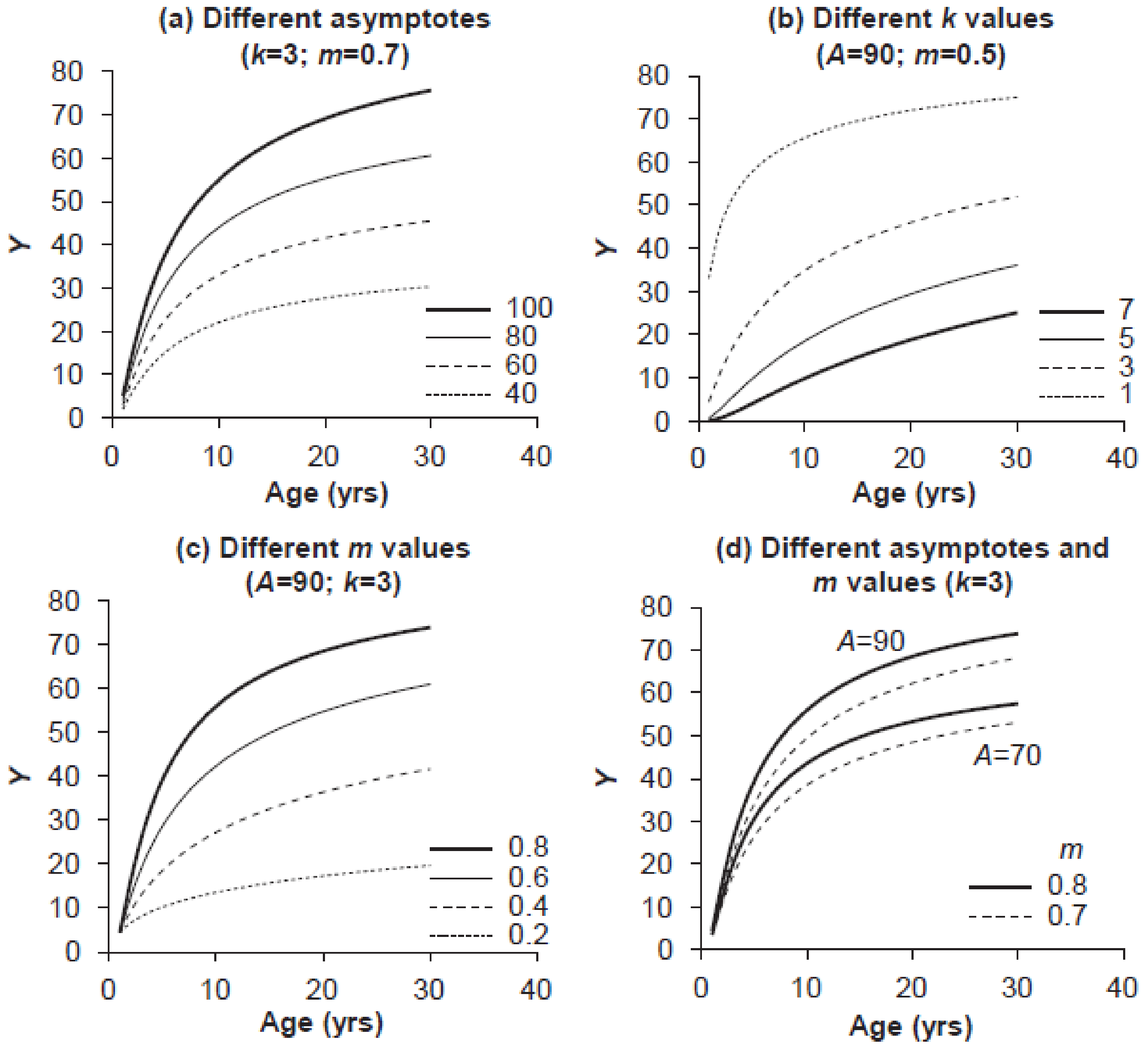
$$Y = A\, e^{-k \frac{1}{t^m}}$$

➔ **The *A* parameter is the assymptote**

➔ **The *k* and *m* parameters are growth rate and shape parameters:**

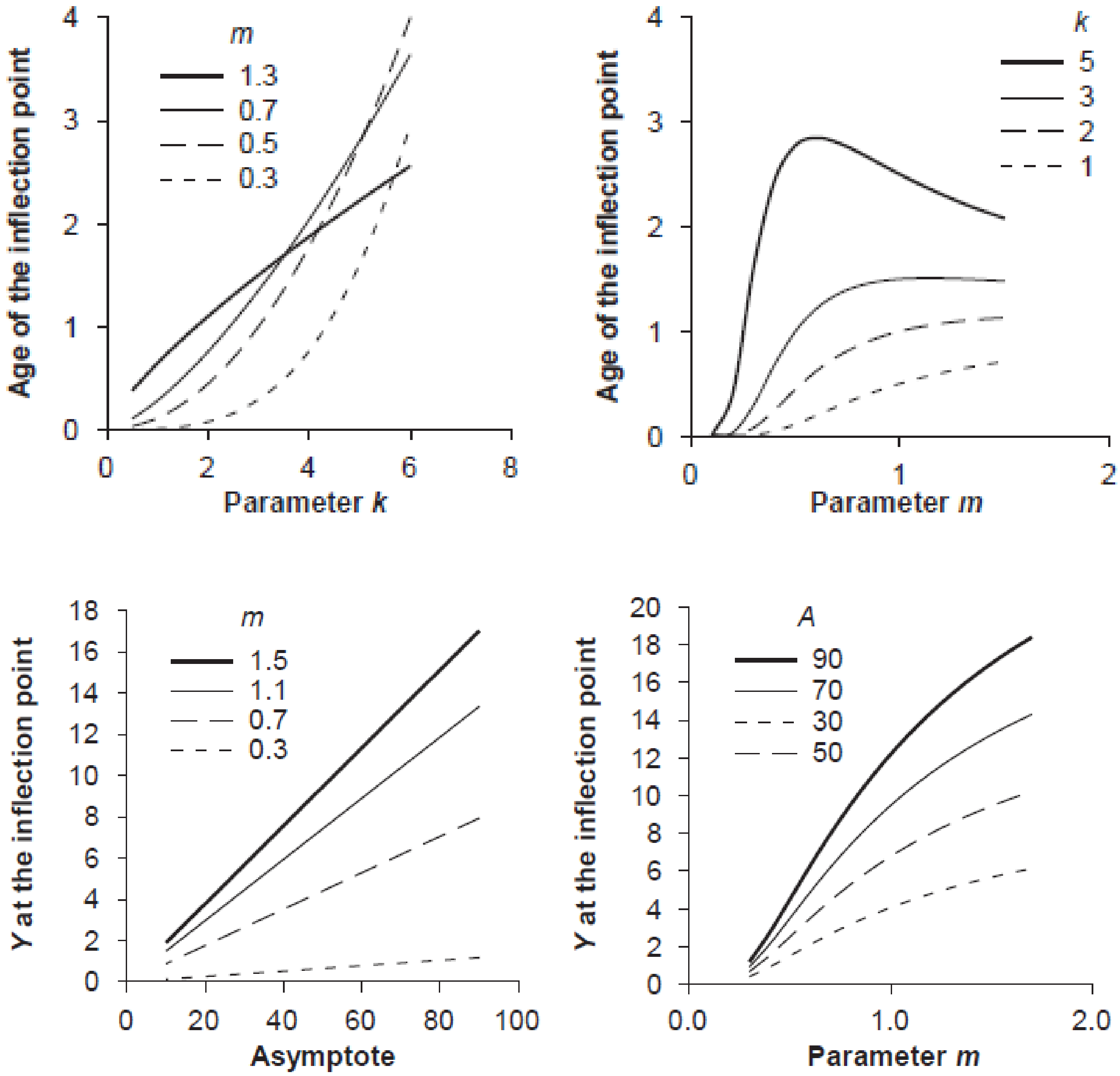
- ***k* is inversely related with the growth rate**
- ***m* influences the age at which the inflexion point occurs**

Lundqvist-Korf type functions



Lundqvist-Korf type functions

Location of the inflection point



Richards type functions

- **Differential form based on the hypothesis that the absolute growth rate of biomass (or volume) is modeled as:**
  - **the *anabolic rate* (construction metabolism), proportional to the photossintethicaly active area (expressed as an allometric relationship with biomass)**
  - **the *catabolic rate* (destruction metabolism), proportional to biomass**

Anabolic rate ———— $c_1 S = c_1\left(c_0 Y^m\right) = c_2 Y^m$

Catabolic rate ———— $c_3 Y$

Growth rate ———— $c_2 Y^m - c_3 Y$

S – photossintethically active biomass ; Y – biomass; m – alometric coefficient; c0,c1,c2,c3 – proportionality coefficients

Richards type functions

- **The differential form of the Richards function is then:**

$$\frac{dY}{dt} = \eta Y^{m} - \gamma Y$$

- **By integration and using the initial condition *$y(t_0)$=0*, the integral form of the Richards function is obtained:**

$$Y = A\left(1 - c\,e^{-k\,t}\right)^{\frac{1}{1-m}}$$

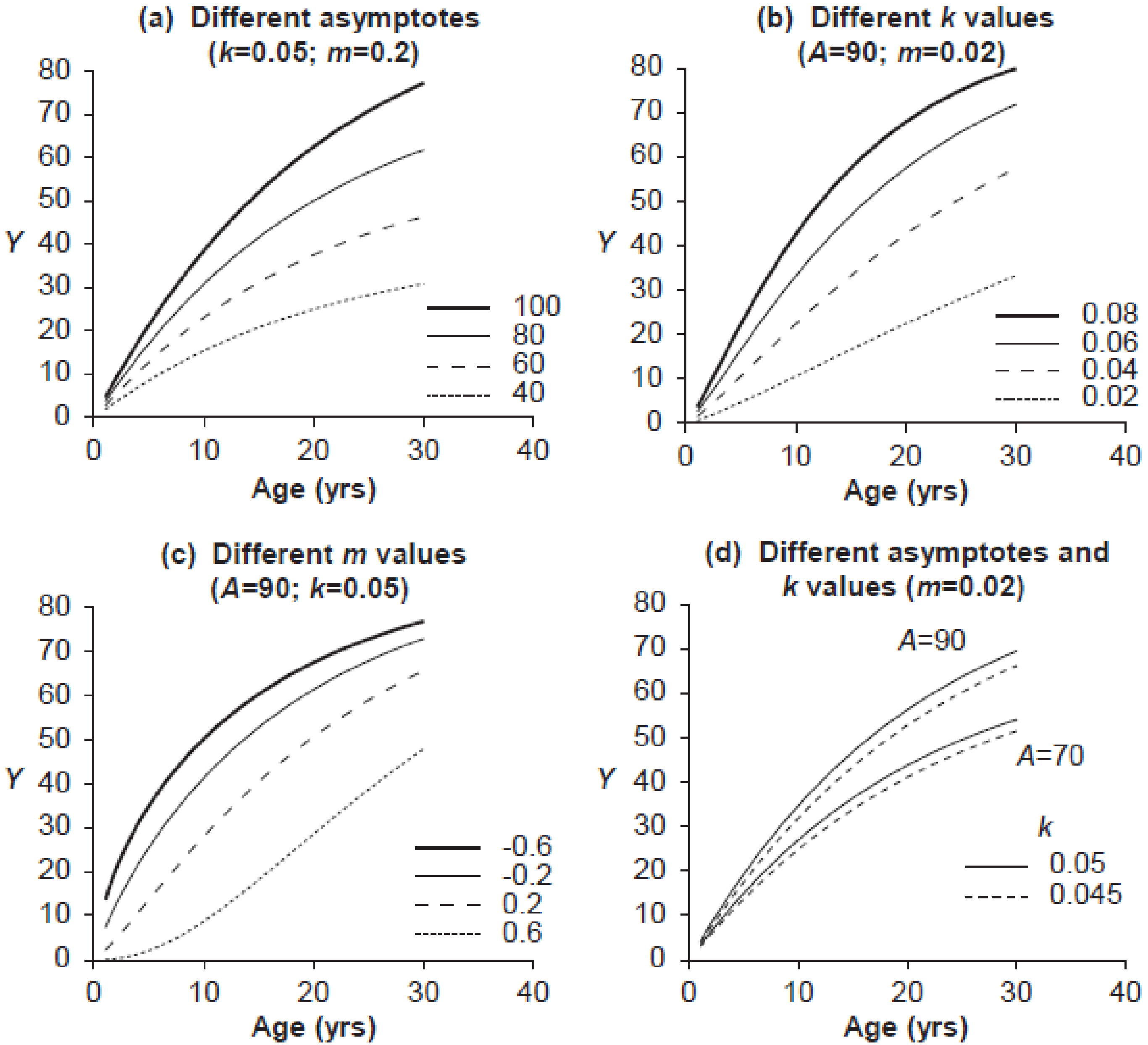
with parameters *m*, *c*, *k* and *A* where:

$$c = e^{-(1-m)\gamma\, t_0} = e^{-k\,t_0}$$

$$k = (1-m)\gamma$$

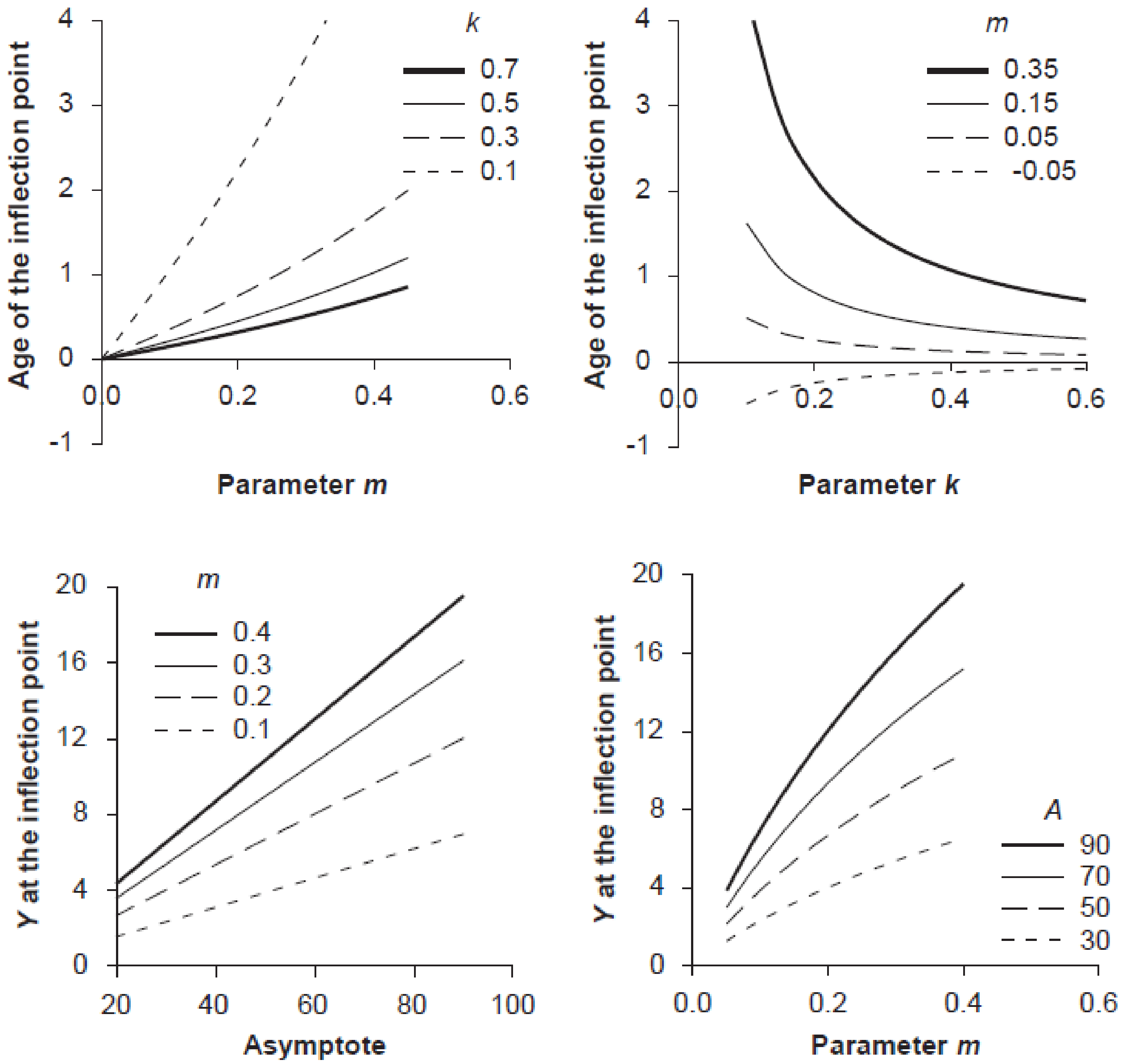
$$A = \left(\frac{\eta}{\gamma}\right)^{\frac{1}{1-m}}$$

Richards function



Richards function

Location of the inflection point



Functions of the Richards type

- **Monomolecular, when the m parameter equal to 0 (no inflection point)**

$$Y = A\left(1 - c\,e^{-k\,t}\right)$$

- **Logistic, when the m parameter equal to 2 (symmetric in relation to the inflection point)**

$$Y = \frac{A}{\left(1 + c\,e^{-k\,t}\right)}$$

- **Generalized logistic**

  ➔ If kt is a function of t, usually a polynomial

- **Gompertz, when the m parameter $\rightarrow 1$**

$$Y = A\,e^{-c\,e^{-k\,t}}$$

Hossfeld IV function

- **The Hossfeld IV function is a sigmoid function, originally proposed in 1822 (Zeide 1993), for the description of tree growth:**

$$Y = \frac{t^k}{c + t^k / A} = A \frac{t^k}{Ac + t^k}$$

- **The function can also be obtained from the generalized logistic by using *f(X,t)=-klog(t)*. Consequently some authors designate it as the log-logistic growth function**

McDill-Amateis function

- **Integral form:**

$$Y = \frac{A}{1-\left(1-\frac{A}{Y_0}\right)\left(\frac{t_0}{t}\right)^k}$$

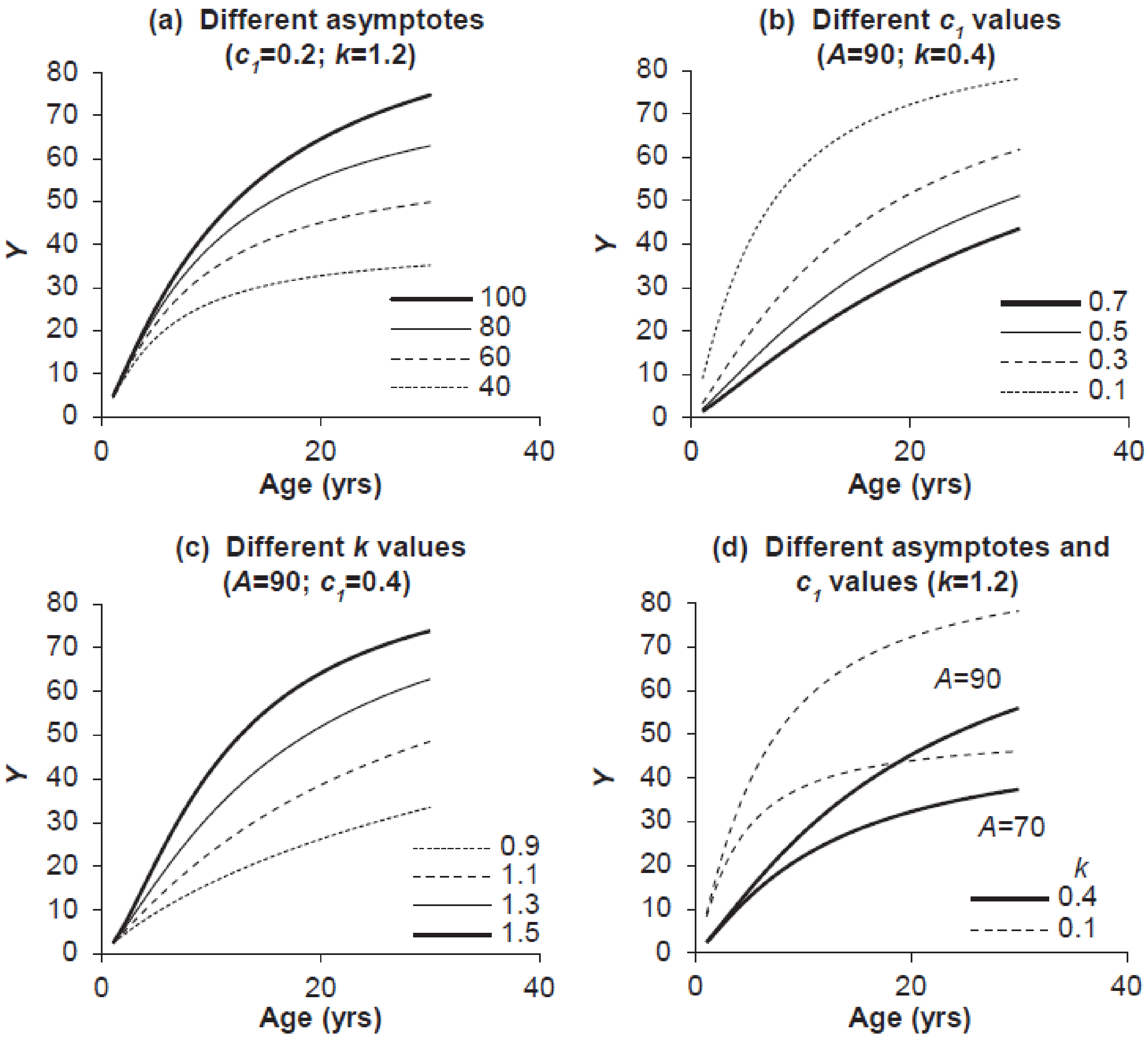
where (t_0, Y_0) is the initial condition and k expresses the growth rate

➔ **By making**

$$c = \left(\frac{1}{Y_0} - \frac{1}{A}\right)t_0^K$$

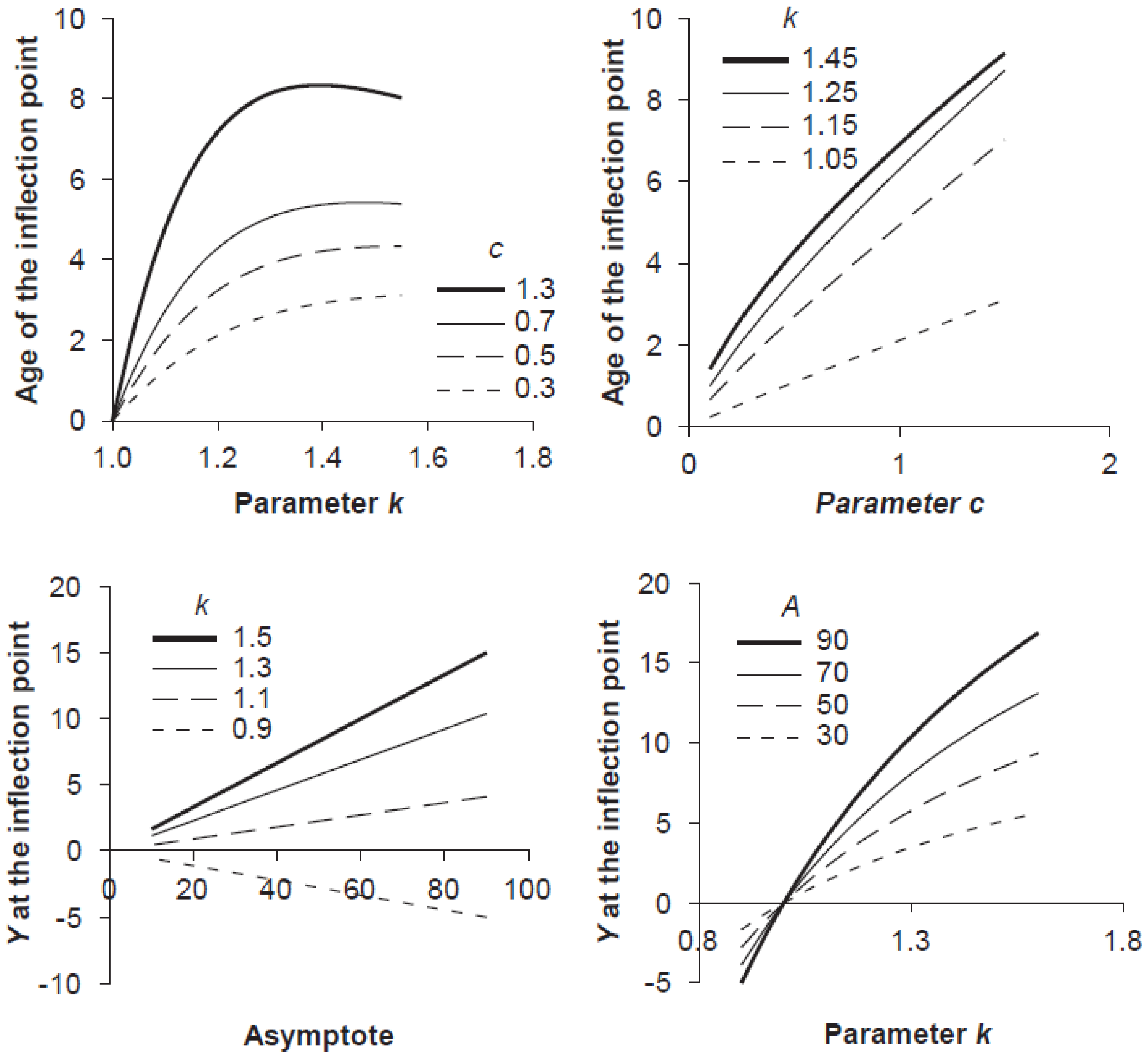
the integral form of the McDill-Amateis function coincides with the Hossfeld IV function

Hossfeld IV function



Hossfeld IV function

Location of the inflection point



- **Zeide decomposition of growth functions**

Zeide decomposition of growth functions

- **Zeide found out that all the growth functions can be decomposed into two componentes (similar o the development of the Richards type functions):**

  ➔ **Growth expansion** - represents the innate tendency towards exponential multiplication and is associated with biotic potential, photosynthetic activity, absorption of nutrients, constructive metabolism, anabolismo

  ➔ **Growth decline** - represents the constraints imposed by external (competition, limited resources, respiration, and stress) and internal (self-regulatory mechanisms and aging) factors

Zeide decomposition of growth functions

- The decomposition can be achieved either by a subtraction or a division (subtraction of logarithms) of the two effects

- All the equations analyzed by Zeide are particular cases of the two following forms:

→ LTD $$\ln y' = k + p\ln y + q\ln t \leftrightarrow y' = k_1 y^p t^q$$

→ TD $$\ln y' = k + p\ln y + qt \leftrightarrow y' = k_1 y^p e^{qt}$$

where $p>0$, $q<0$ and $k=e^k$

- In both forms the expansion component is proportional to ln(y) or, in the antilog form, is a power of size

- In LTD the decline component is proportional to the ln of age while in TD it is proportional to age

Zeide decomposition of growth functions

- Zeide proposed a third form in which the declining component is expressed as a function of size instead of age:

$$\ln y' = k + p \ln y + q y \leftrightarrow y' = k_1 y^p e^{qy}$$

- The three forms are very useful for the direct modeling of tree and/or stand growth - these forms provide some assurance that the resulting model will display appropriate behavior form a biological stand point

- **Simultaneous modeling of several individuals (Families of growth functions)**

Families of growth functions

- The fitting of a growth function to data from a permanent plot is straightforward

  Example:

  → Fitting the Lundqvist function to basal area and dominant height growth data from a permanent plot

$$Y = A\, e^{-k \frac{1}{t^m}}$$

A - asymptote

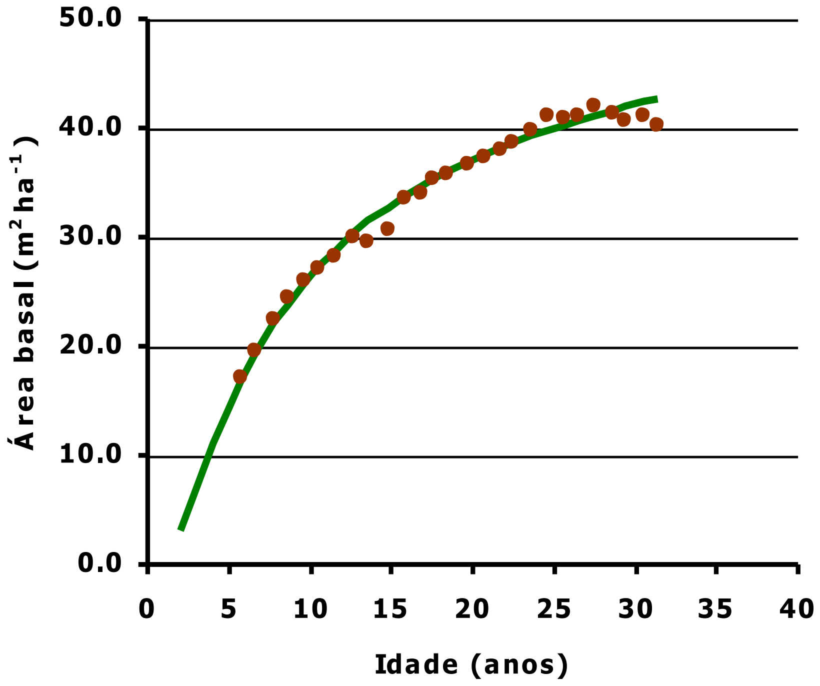
k, *m* – shape parameters

Growth functions

<u>Basal area</u>

A = 58.46, k = 5.13, m = 0.81

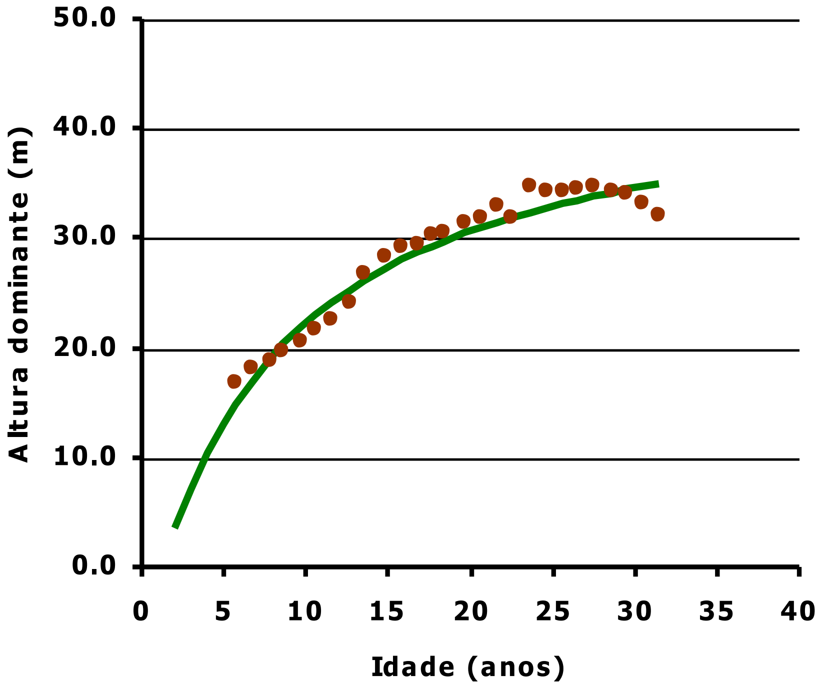
Modelling efficiency = 0.995



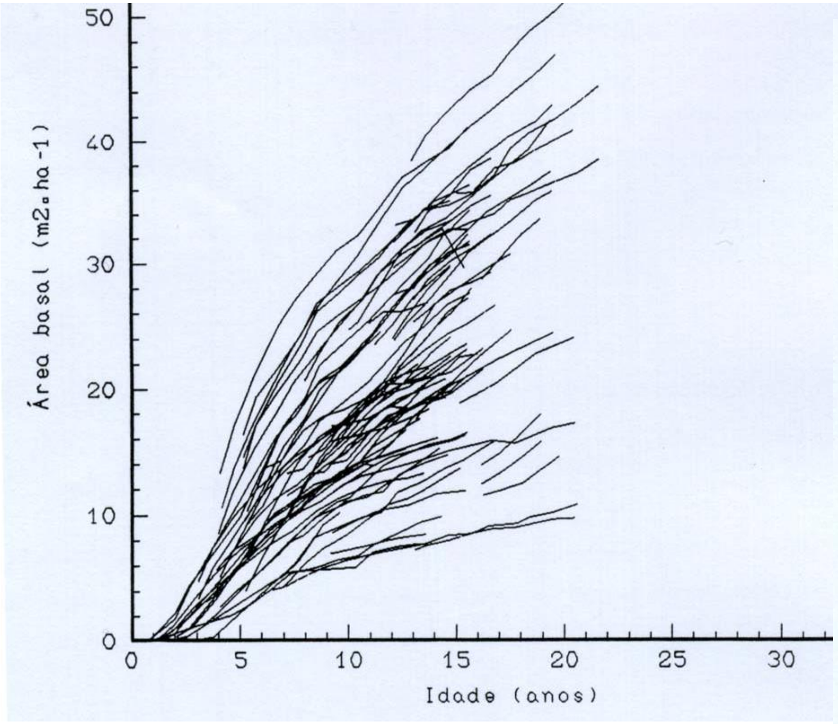
<u>Dominant height</u>

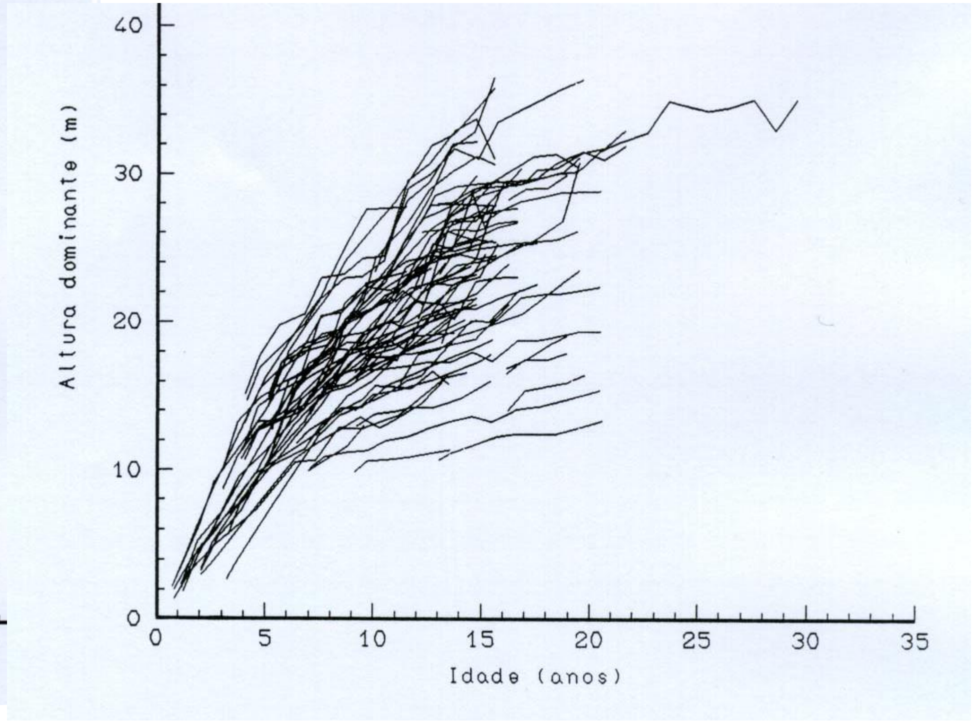
A = 48.75, k = 4.30, m = 0.75

Modelling efficiency = 0.960



▪**But how to model the growth of a series of plots? This is our objective when developing FG&Y models…**





Those plots represent "families" of curves

But how to model the growth of several plots?

- There are several methods to simultaneously model the growth of several plots:

  - → Using growth functions formulated as difference equations – ADA
  - → Expressing the parameters as a function of site and/or tree/stand variables
  - → Using growth functions formulated as difference equations – GADA
  - → Using mixed models

- In this course we will just focus the two first methods (information on other methods available at the end of the slides)

Using growth functions formulated as difference equations - ADA

- Algebraic difference approach (ADA)

  ➔ When formulating a growth function as a difference equation, it is assumed that the curves belonging to the same "family" differ just by one parameter - the free parameter

  ➔ A growth function with 3 parameters allows for 3 different formulations, usually denoted by the free parameter

  ➔ For example for the Richards function:

    Richards-A (model with site specific asymptote)

    Richards-k (model with common asymptote)

    Richards-m (model with common asymptote)

Using growth functions formulated as difference equations - ADA

Example with the Lundqvist function, formulation with common asymptote and common n parameter, *A* as free parameter (Kundqvist-A):

$$Y = A\, e^{-k\frac{1}{t^m}} \Rightarrow A = \frac{Y}{e^{-k\frac{1}{t^m}}} \longrightarrow \frac{Y_2}{e^{-k\frac{1}{t_2^m}}} = \frac{Y_1}{e^{-k\frac{1}{t_1^m}}}$$

$$Y_2 = Y_1\, e^{-k\left(\frac{1}{t_2^m} - \frac{1}{t_1^m}\right)} \Leftrightarrow Y = Y_0\, e^{-k\left(\frac{1}{t^m} - \frac{1}{t_0^m}\right)}$$

A specific curve of the family is defined by the value of the free parameter

In practice, the free parameter is a function of an initial condition (Y_0,t_0)

Expressing parameters as a function of tree/stand variables

- Example with the Lundqvist function fit to basal area growth of eucalyptus (GLOBULUS 2.1 model) :

$$G = A \ e^{-k \ \left(\frac{1}{t}\right)^{m}}$$

$$A = A \ S^{2}$$

$$k = k + k \ S + k \ \frac{Npl}{1000} + k \ fe \qquad \text{with} \quad fe = \frac{100}{S\sqrt{Npl}}$$

$$m = m + m \ \ln(S) + m \ \frac{N}{1000}$$

Difference equations when the parameters are expressed as a function of tree/stand variables

- Example with the Lundqvist function fit to basal area growth of eucalyptus with k as free parameter:

$$G = A\ e^{-k\left(\frac{1}{t}\right)^{(m_0 + m_1 N)}}$$

- By using the same method as above we obtain:

$$G_2 = A\left(\frac{G_1}{A}\right)^{\frac{t_1^{m_0 + m_1 N_1}}{t_2^{m_0 + m_1 N_2}}}$$

Using growth functions formulated as difference equations - GADA

- Generalized algebraic difference approach (GADA)

  - ➔ One of the problems with ADA is the fact that it originates formulations that differ just by one parameter
  - ➔ With GADA it is possible to obtain formulations that have more than one site-specific parameter
  - ➔ In GADA parameters are assumed to be function of an unobservable set of variables (denoted by X) that express site differences
  - ➔ The equations are then solved by X, which, for a particular site, is substituted in the original equation (X_0)

Using growth functions formulated as difference equations - GADA

- ✓ Example with the Schumacher function:

$$\ln(Y) = \alpha + \frac{\beta}{t}$$

Suppose that α=X and β=γX, then

$$\ln(Y) = X + \frac{\gamma X}{t} \longrightarrow X = \frac{\ln(Y)}{1+\gamma/t} \longrightarrow X_0 = \frac{\ln(Y_0)}{1+\gamma/t_0}$$

By substituting X0 into the previous expression, we get

$$\ln(Y) = \ln(Y_0)\frac{t_0(t-\gamma)}{t(t_0-\gamma)}$$

Using growth functions formulated as difference equations - GADA

- ✓ Another example with the Schumacher function
  Suppose now that α=X and β=X, then

$$\ln(Y) = X - \frac{\beta}{t} \quad \text{and} \quad \ln(Y) = \alpha - \frac{X}{t} \longrightarrow 2\ \ln(Y) = \left(X - \frac{\beta}{t}\right) + \left(\alpha - \frac{X}{t}\right)$$

- ✓ Solving for X:

$$X = \frac{t[\ln(Y) - \alpha] + \beta}{t - 1} \longrightarrow X_0 = \frac{t_0[\ln(Y_0) - \alpha] + \beta}{t_0 - 1}$$

- ✓ Finally, substituting X0 in the previous expression

$$\ln(Y) = \alpha - \frac{\beta}{t} + \frac{(t-1)t_0}{(t_0 - 1)t}\left[\ln(Y_0) - \alpha + \frac{\beta}{t_0}\right]$$

Using mixed-models

- Mixed-models (linear and non-linear) “split” the model error according to different sources of variation, such as:
  - ➔ Region
  - ➔ Stand
  - ➔ Plots
  - ➔ …
- When using a model fitted with mixed-models theory it is possible to calibrate the parameters with random components by measuring a small sample of individuals
- This means that it is possible to use specific parameters for a particular tree/stand

Which is the best method to model “families” of growth functions?

- There is no best method to model “families” of growth functions
- If appropriate the three methods can be combined in order to obtain more flexible growth models

- **Formulating growth functions without age explicit**

Formulating growth functions without age explicit

- In many applications age is not known, e.g. in trees that do not exhibit easy to measure growth rings or in uneven aged stands

- For these cases it is useful to derive formulations of growth functions in which age is not explicit

- The derivation of these formulations is obtained by expressing t as a function of the variable and the parameters and substituting it in the growth function written for t+a (Tomé et al. 2006)

Formulating growth functions without age explicit

✓ Example with the Lundqvist function:

$$Y_t = A\, e^{-k \frac{1}{t^m}} \quad \longrightarrow \quad t = \left[\frac{-k}{ln(y_t / A)} \right]^{\frac{1}{m}}$$

$$Y_{t+a} = A\, e^{-k \frac{1}{(t+a)^m}} \quad \longrightarrow \quad Y_{t+a} = A\, e^{-k \frac{1}{\left(\left[\frac{-k}{ln(y_t/A)} \right]^{\frac{1}{m}} + a \right)^m}}$$