Principles of Linear Programming: Sensitivity analysis

11 May 2016

Keeping the river clean

▷ Sensitivity analysis

A pulp mill makes mechanical and chemical pulp and during the production process it pollutes the river in which it spills its spent waters. The owners would like to minimize pollution, keeping at least 300 people employed at the mill and generating at least 40000€ of revenue per day.

Keeping the river clean

- The maximum capacity of the mill is 300 tons per day to make mechanical pulp and 200 tons per day to make chemical pulp (the mechanical pulp line cannot be used to make chemical pulp, and vice-versa)
- Both mechanical and chemical pulp require the labor of 1 worker for about 1 day, or 1 workday (wd), per ton produced
- Pollution is measured by the biological oxygen demand (BOD). 1 ton of mechanical pulp produces 1 unit of BOD, 1 ton of chemical pulp produces 1.5 units
- The chemical pulp sells at 200€, the mechanical pulp at 100€per ton.

Keeping the river clean

x_1 - Amount of mechanical pulp produced (t/d)

x_2 - Amount of chemical pulp produced (t/d).

$$\min\ Z = x_1 + 1.5x_2 \quad \text{units of BOD/d} \tag{1}$$

$$subject\ to$$

$$x_1 + x_2 \geq 300 \quad \text{workers employed} \tag{2}$$

$$100x_1 + 200x_2 \geq 40000 \quad \text{revenue, euros/d} \tag{3}$$

$$x_1 \leq 300 \quad \text{mechanical pulping capacity, t/day} \tag{4}$$

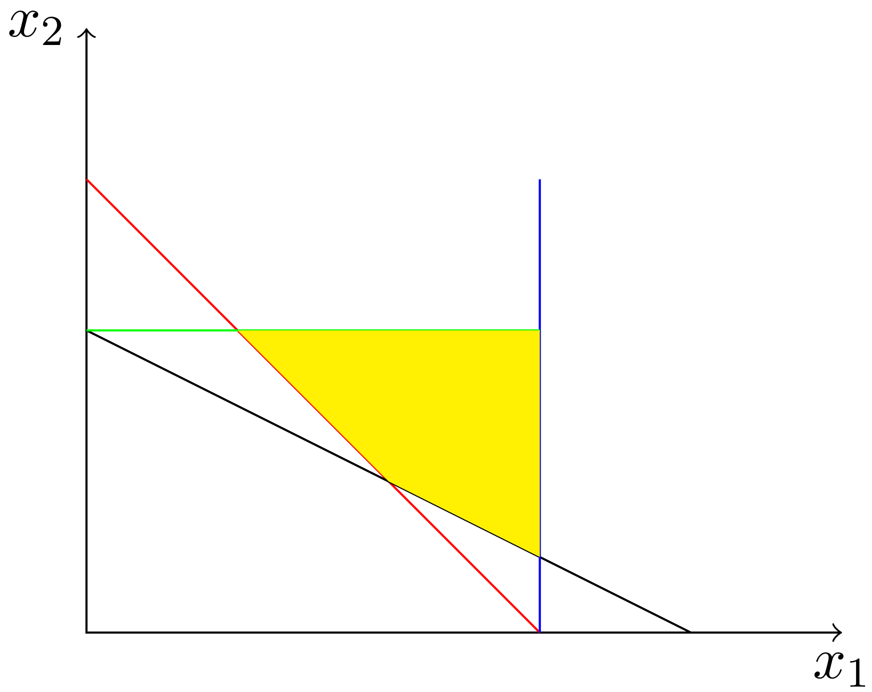
$$x_2 \leq 200 \quad \text{chemical pulping capacity, t/day} \tag{5}$$

$$x_1, x_2 \geq 0 \tag{6}$$

Optimal solution $(x_1^*, x_2^*) = (200, 100) \longrightarrow Z^* = 350$

Keeping the river clean

▷ Sensitivity analysis



Level set k,
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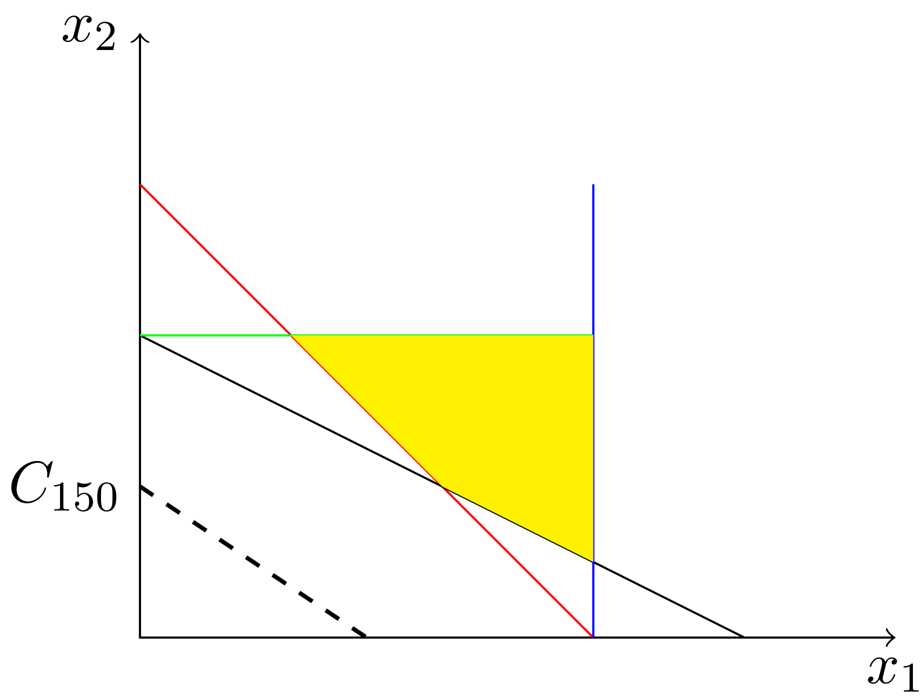

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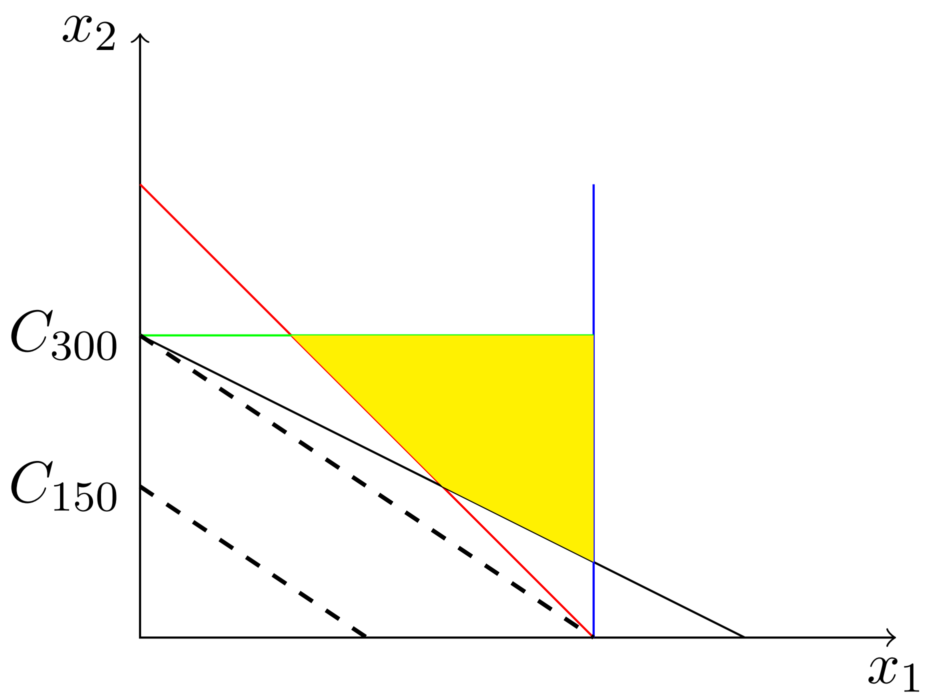

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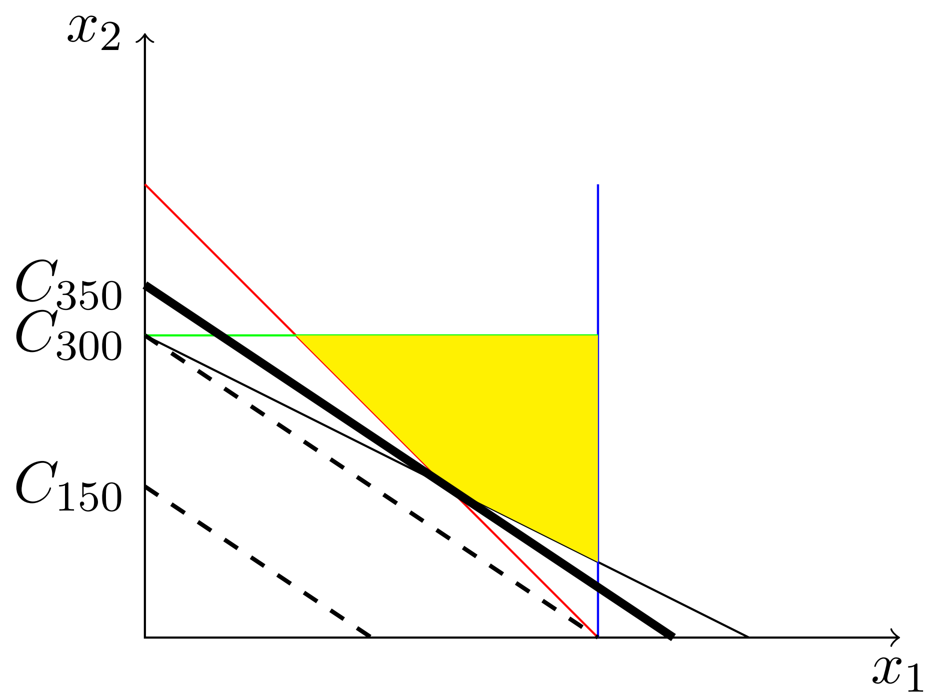

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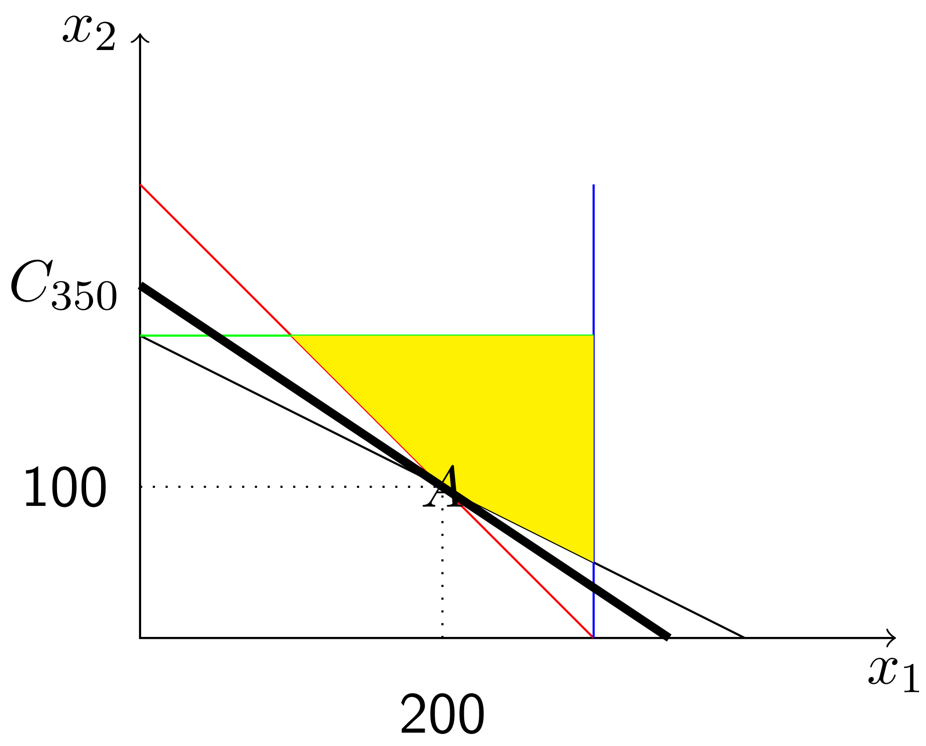


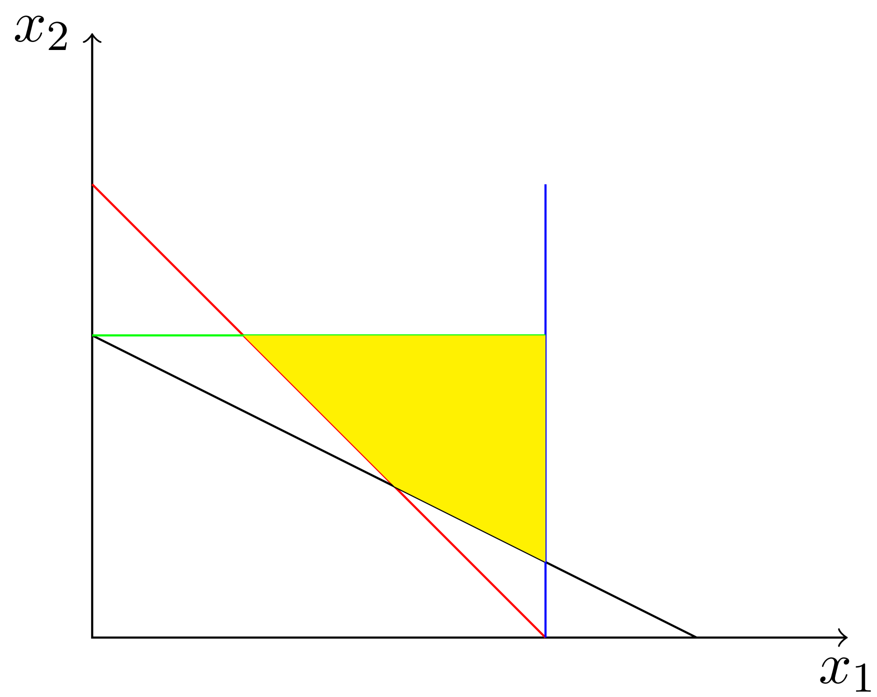
Optimal solution $A = (200, 100) \longrightarrow Z^* = 350$

Range of optimality for the objective function coefficients

Question: What range of values can the objective function coefficients on x_1 and x_2 be without changing the optimal solution (200,100)?

Range of optimality for the objective function coefficients

▷ Sensitivity analysis



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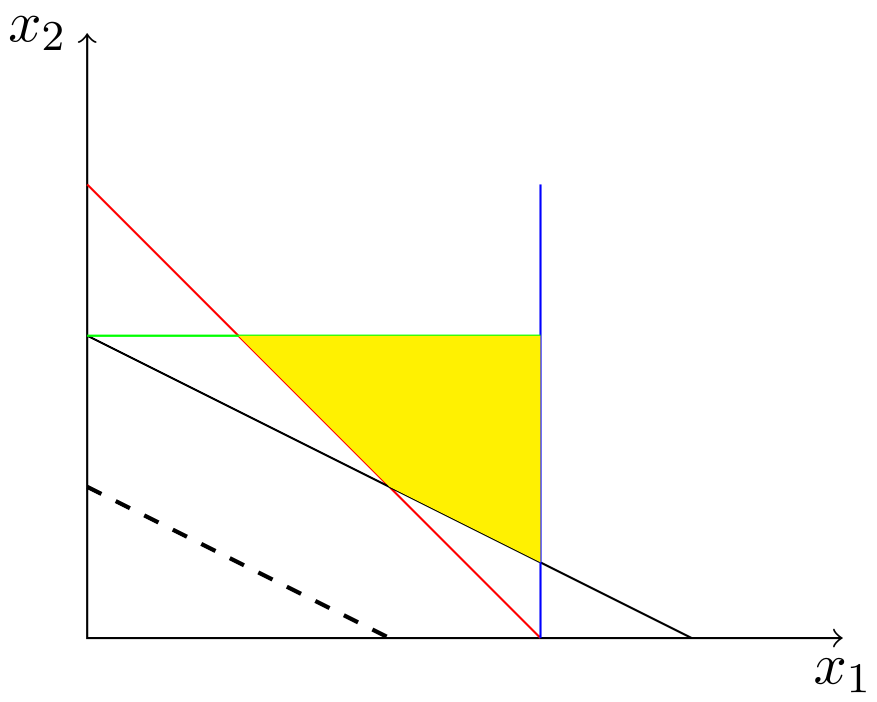

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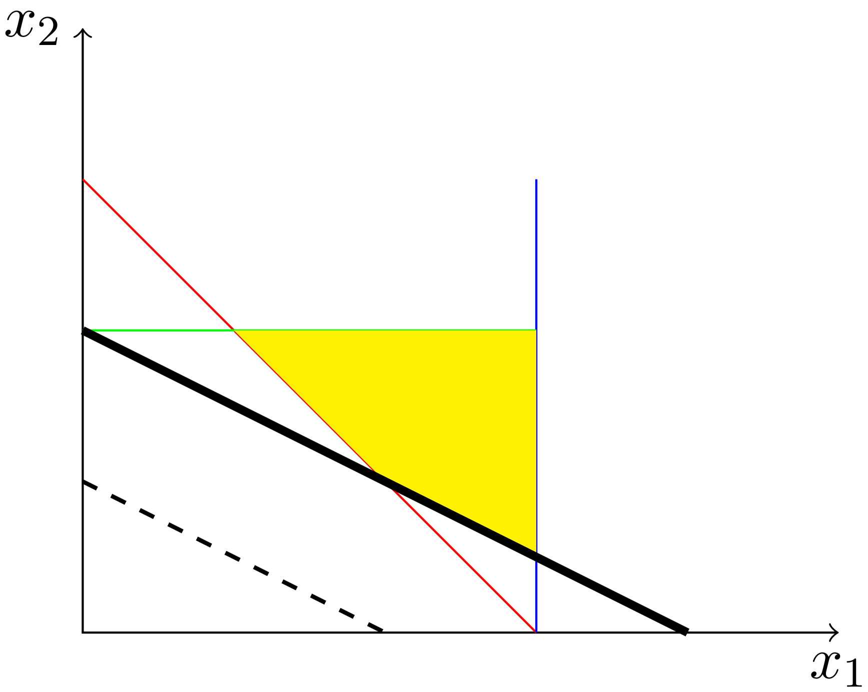

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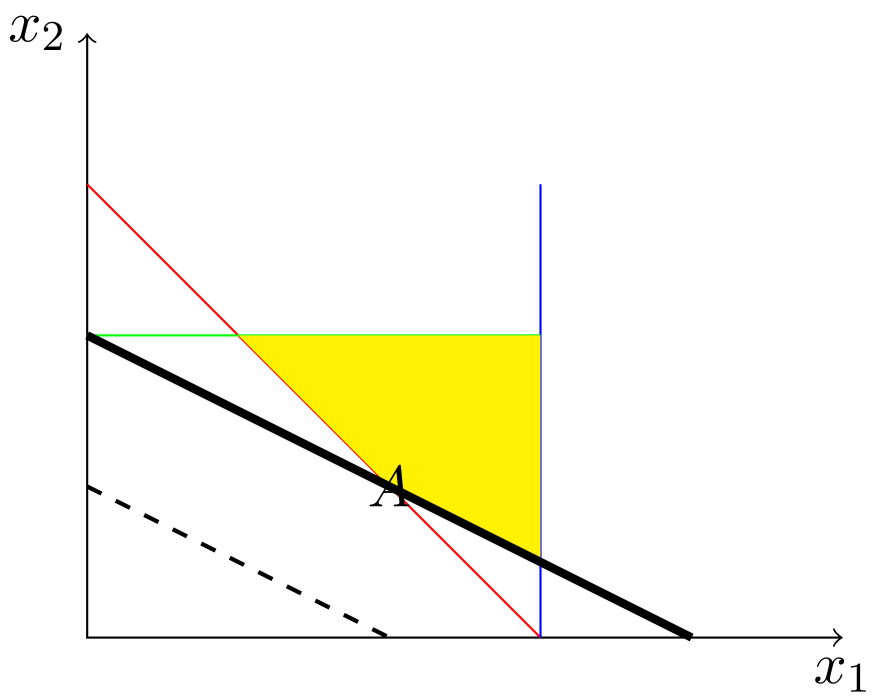

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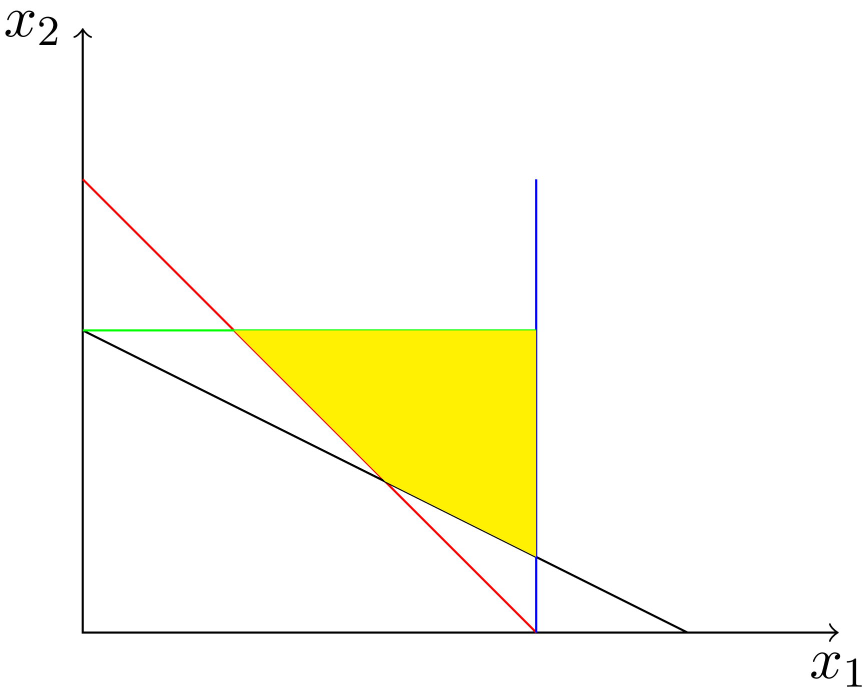

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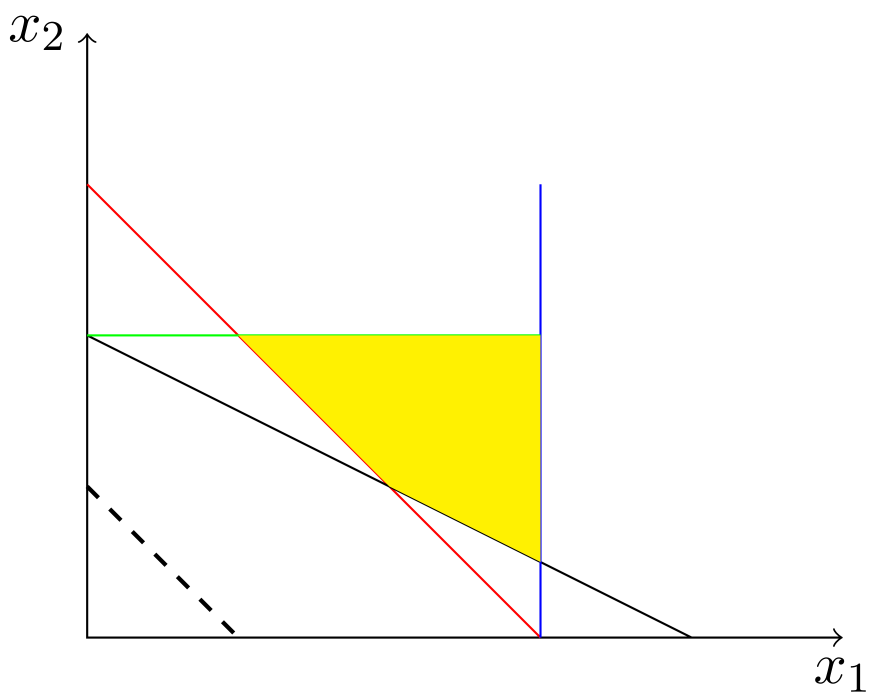

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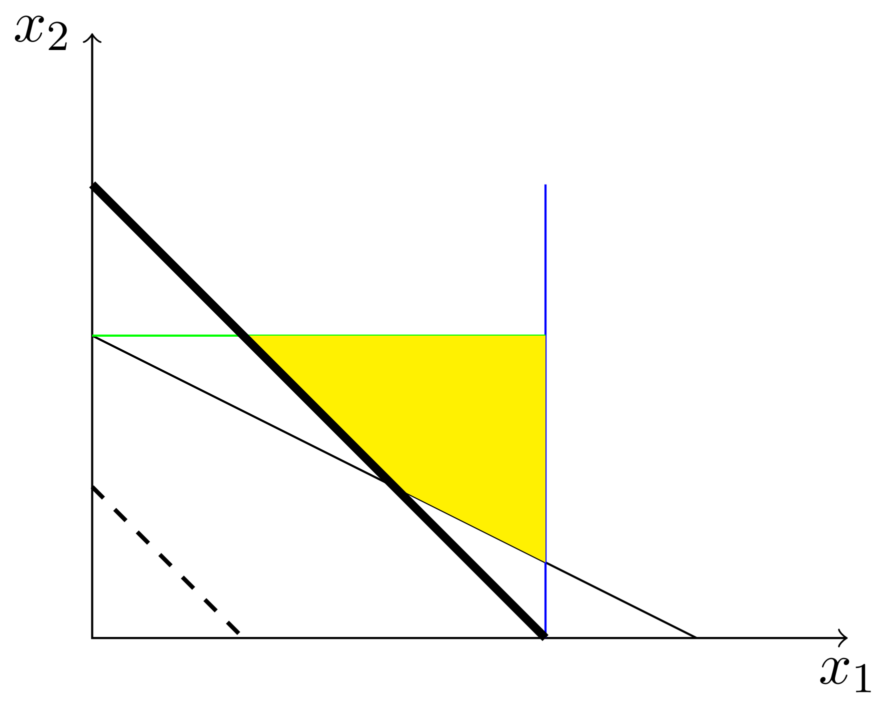

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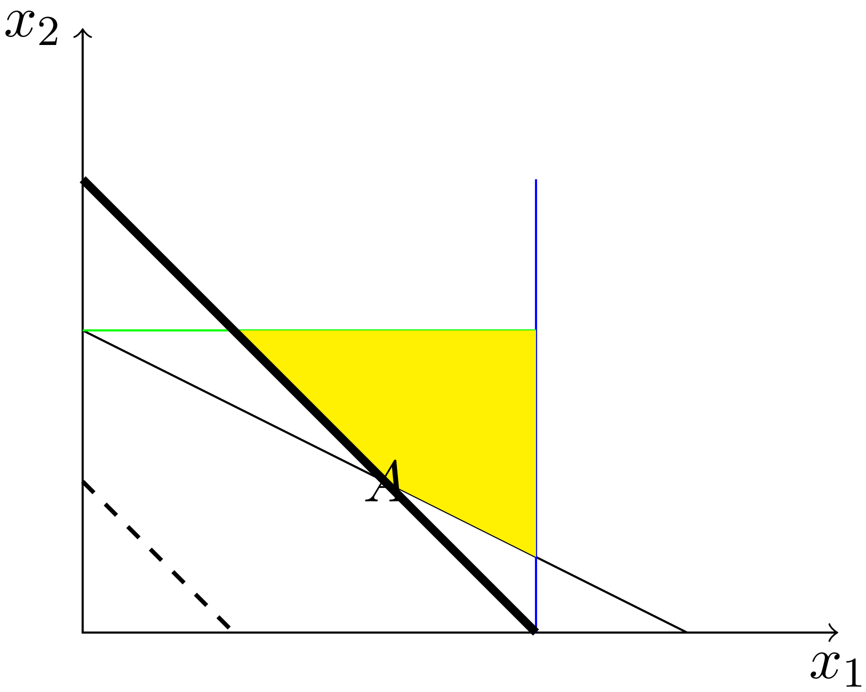

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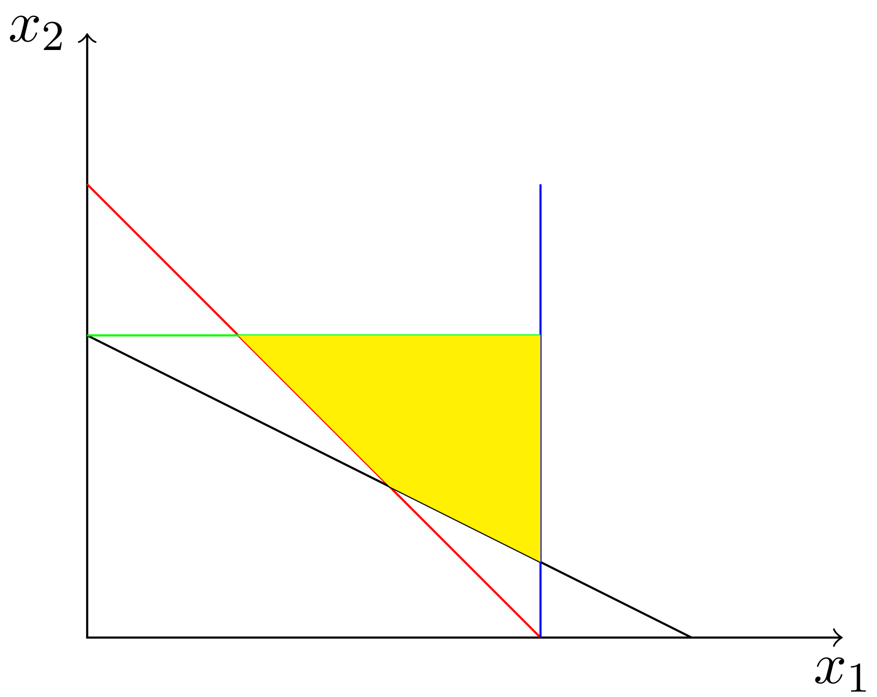

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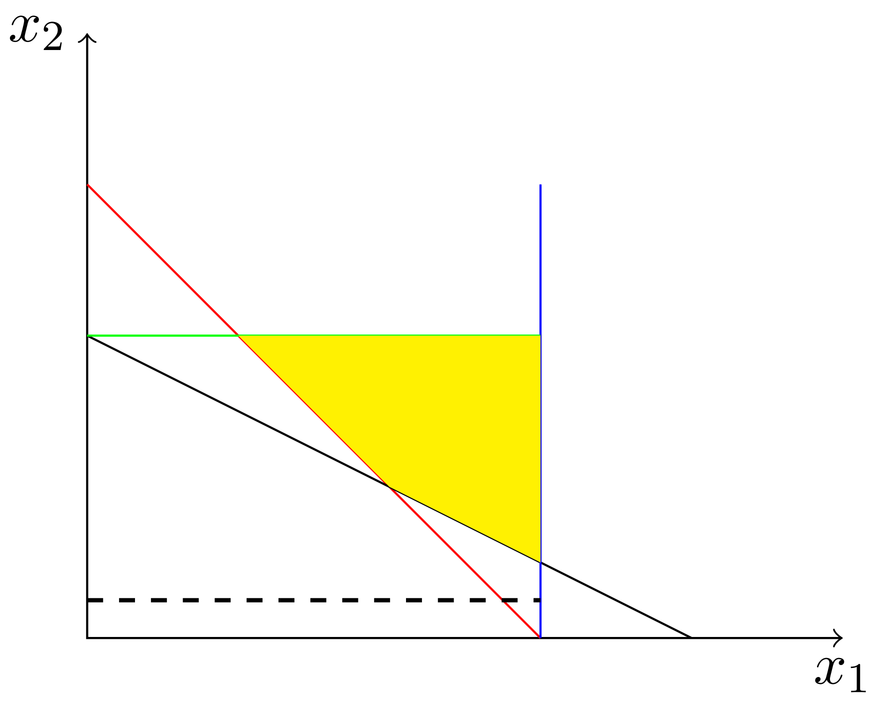

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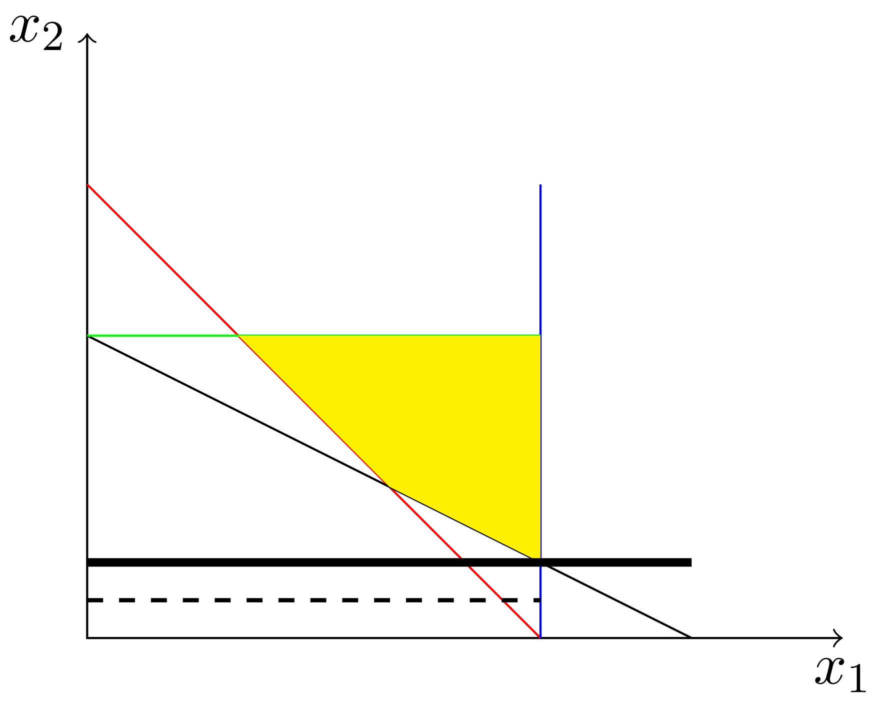

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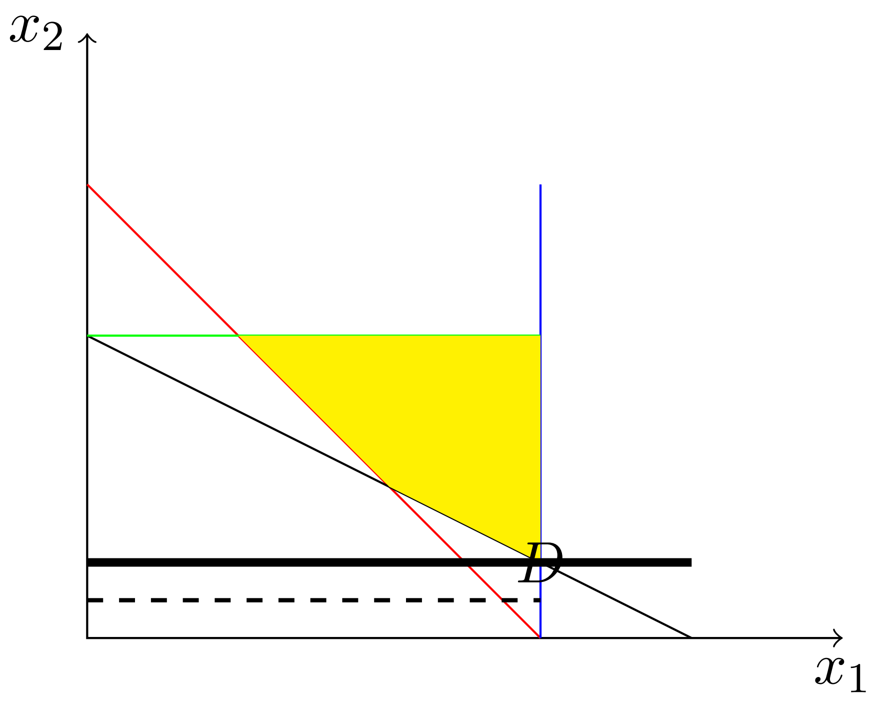

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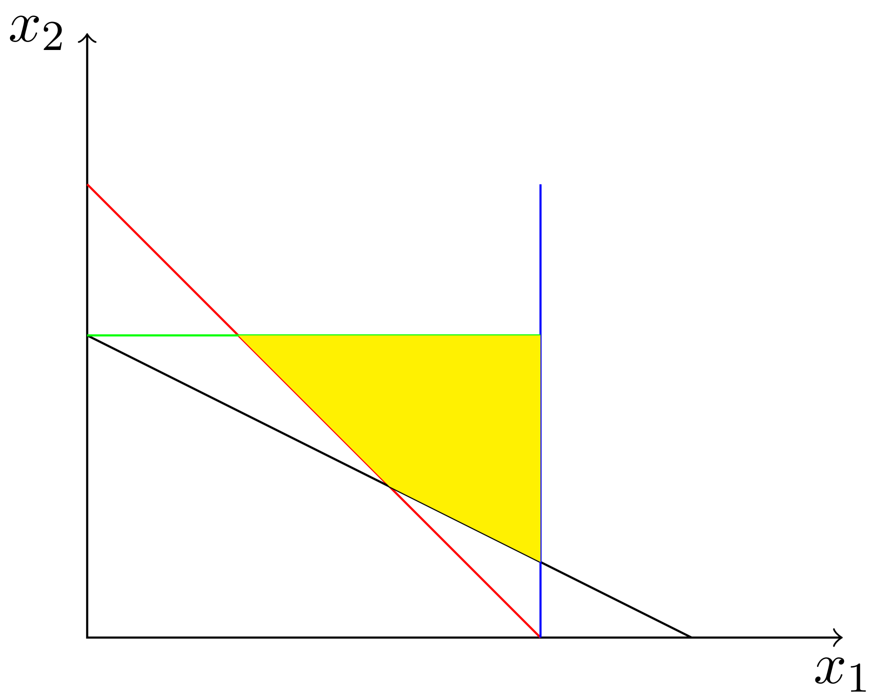

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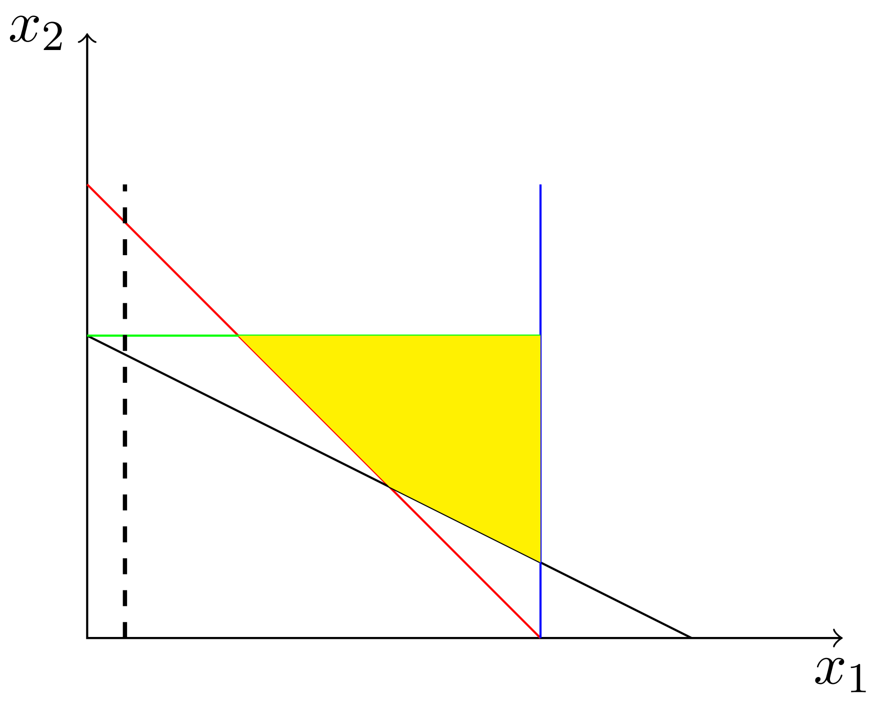

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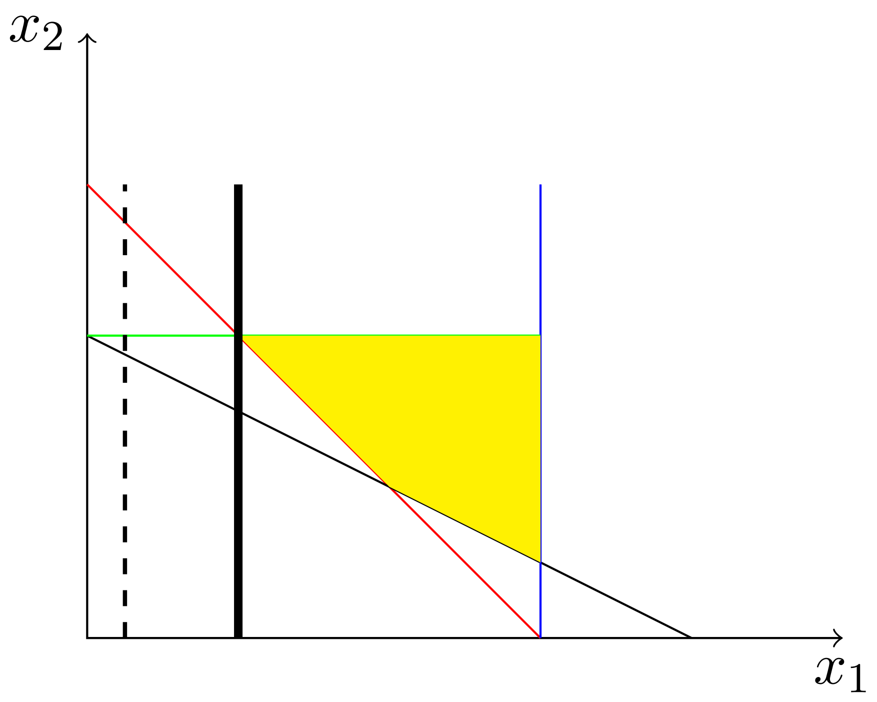

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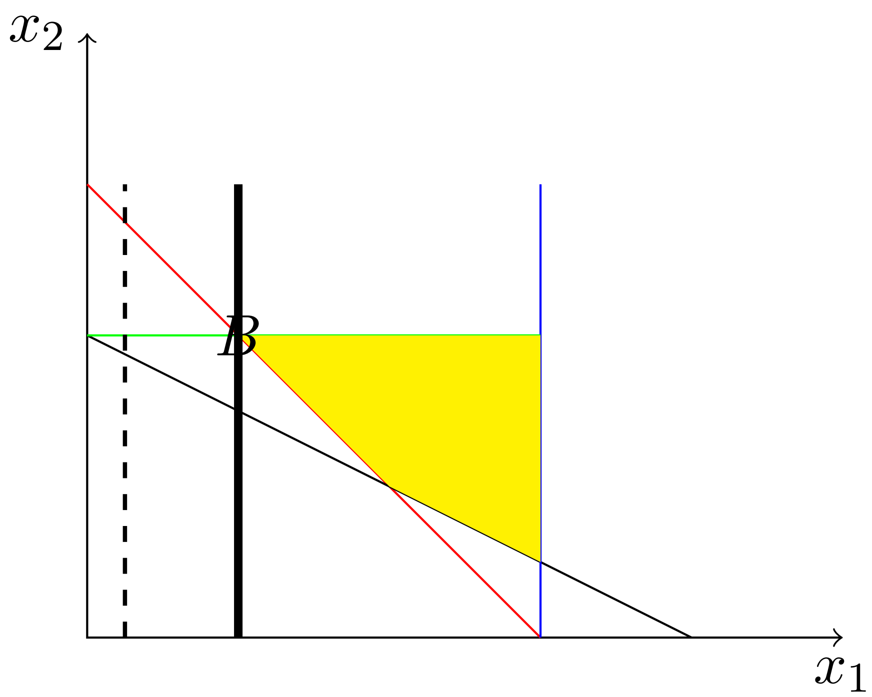

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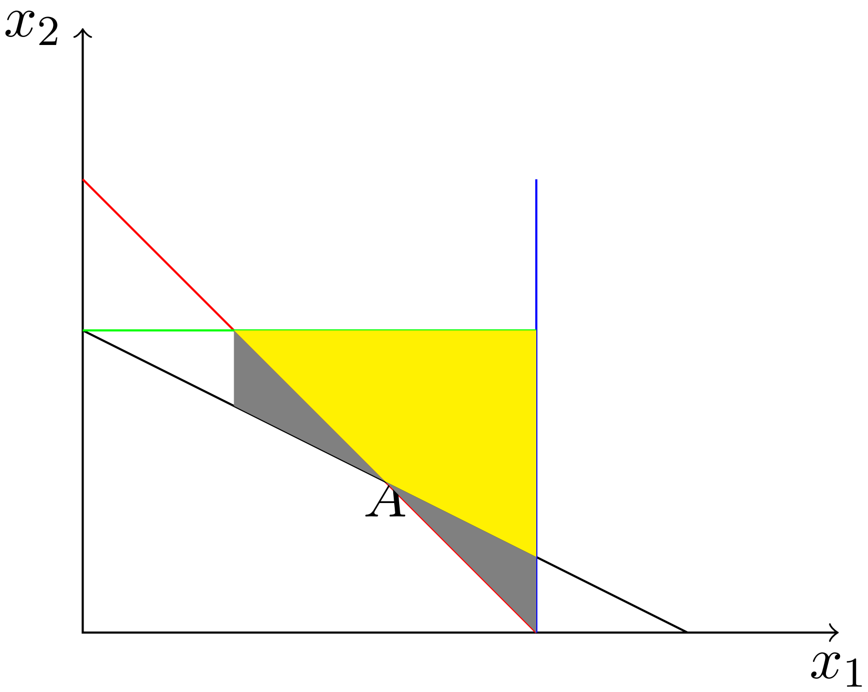


As long as the slope of the set levels is between the slope of constraint (2) and the slope of constraint (3) the optimal solution will always occur at extreme point A.

Range of optimality for the objective function coefficient on x_1

$x_1 + x_2 = 300 \Leftrightarrow x_2 = 300 - x_1$ slope: -1

$100x_1 + 200x_2 = 40000 \Leftrightarrow x_2 = 200 - \frac{1}{2}x_1$ slope: $-\frac{1}{2}$

$c_1 x_1 + 1.5x_2 = k \Leftrightarrow x_2 = \frac{k}{1.5} - \frac{c_1}{1.5}x_1$ slope: $-\frac{c_1}{1.5}$

$$-1 \leq -\frac{c_1}{1.5} \leq -\frac{1}{2} \Leftrightarrow 1.5 \geq c_1 \geq \frac{1.5}{2}$$

Range of optimality: $0.75 \leq c_1 \leq 1.5$

Range of optimality for the objective function coefficients

Question: What range of values can the objective function coefficients on x_1 and x_2 be without changing the optimal solution (200,100)?

As long as the BOD of mechanical pulp is between 0.75 and 1.5, given that the BOD of chemical pulp is 1.5

Note: the optimal value of the objective function will change!

Range of optimality for the objective function coefficients

Question: What range of values can the objective function coefficients on x_1 and x_2 be without changing the optimal solution (200,100)?

As long as the BOD of mechanical pulp is between 0.75 and 1.5, given that the BOD of chemical pulp is 1.5

Note: the optimal value of the objective function will change!

As long as the BOD of chemical pulp is between 1 and 2, given that the BOD of mechanical pulp is 1

Right-hand-side values

Question: How the objective function changes with slight changes of the RHS?

The **shadow price** of a constraint measures the impact on the optimal objective value with the (slight) increase of the RHS.

For example: keeping at least 301 people employed instead of 300?

$$\min Z = x_1 + 1.5x_2 \tag{7}$$

$$subject\,to$$

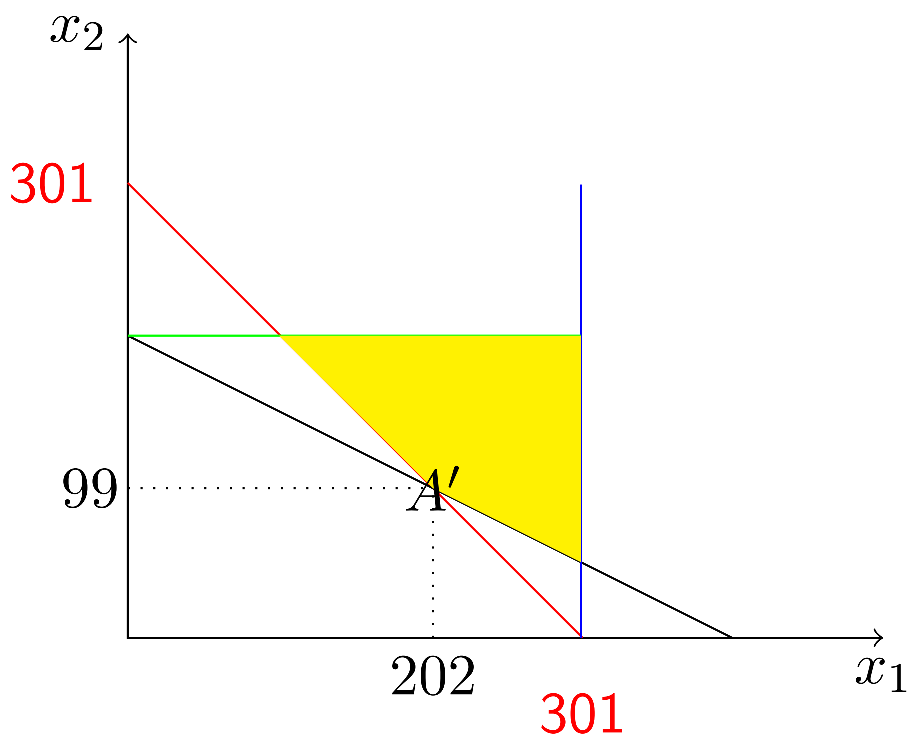
$$\boldsymbol{x_1 + x_2 \geq 301} \tag{8}$$

$$100x_1 + 200x_2 \geq 40000 \tag{9}$$

$$x_1 \leq 300 \tag{10}$$

$$x_2 \leq 200 \tag{11}$$

$$x_1, x_2 \geq 0. \tag{12}$$



Optimal solution $A' = (202, 99) \longrightarrow Z^* = 350.5$

Right-hand-side values

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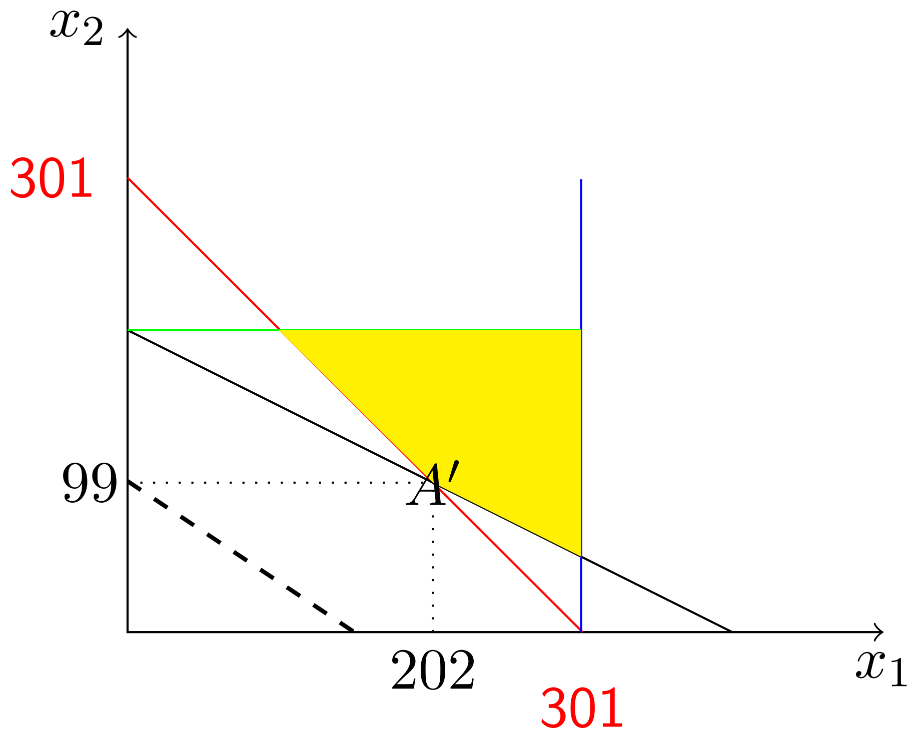

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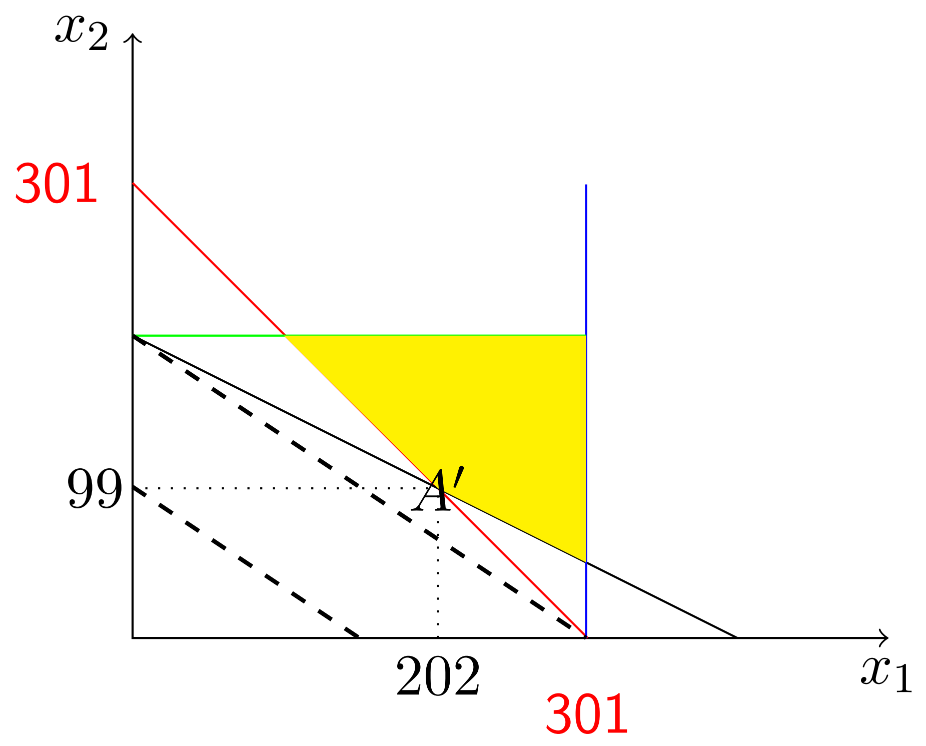

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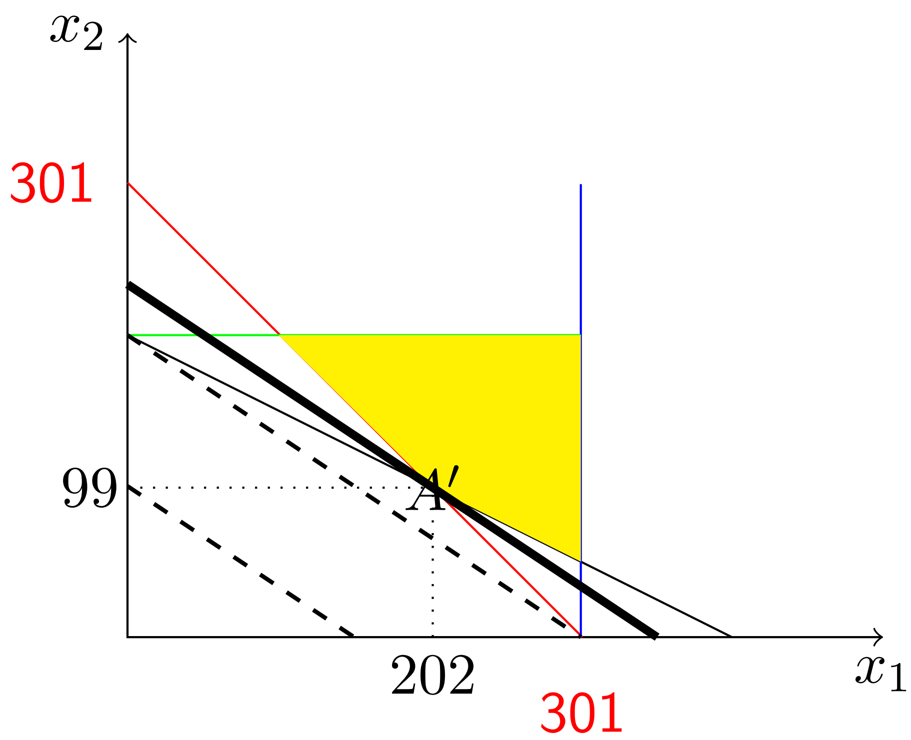


Optimal solution $A' = (202, 99) \longrightarrow Z^* = 350.5$

Question: How the objective function changes with slight changes of the RHS?

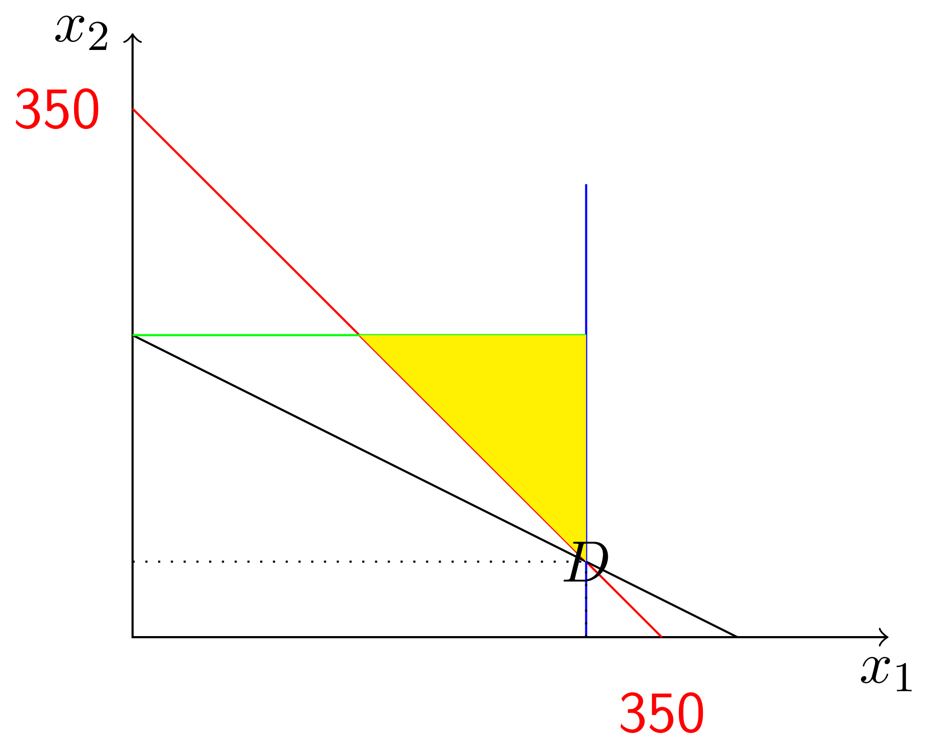
The pollution would increase by 350.5-350=0.5 units of BOD per day for each worker that the mill might employ, for the actual minimum number of workers and minimum revenue required and the maximum capacities for mechanical and chemical pulps.

The shadow price of constraint (2) is 0.5 units of BOD/day/worker.

The shadow price of constraint (2) is valid either with the increase up to 350 workers or to decrease up to 200 workers.

keeping at least 350 people employed instead of 300

▷ Sensitivity analysis

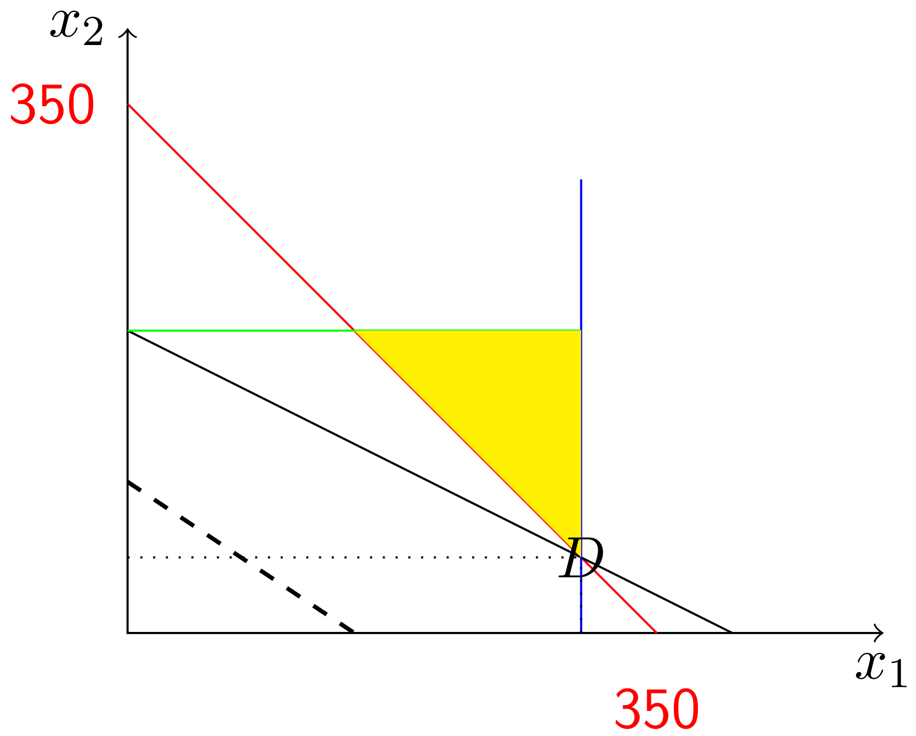


Optimal solution $D = (300, 50) \longrightarrow Z^* = 375$

$$\frac{375 - 350}{50} = 0.5 \text{ units of BOD/d/worker}$$

keeping at least 350 people employed instead of 300

▷ Sensitivity analysis



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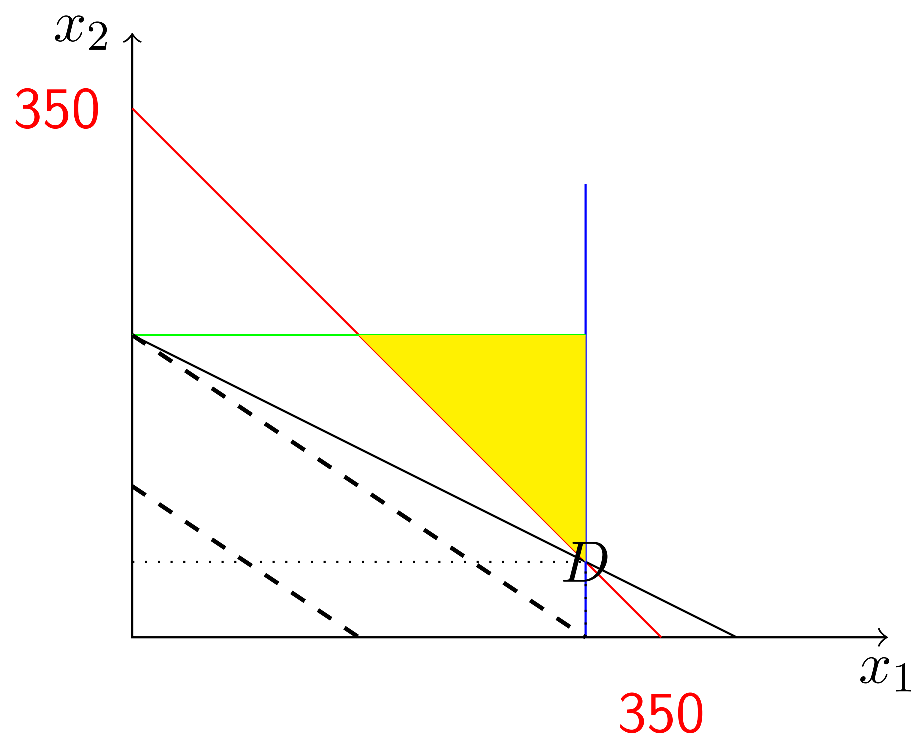

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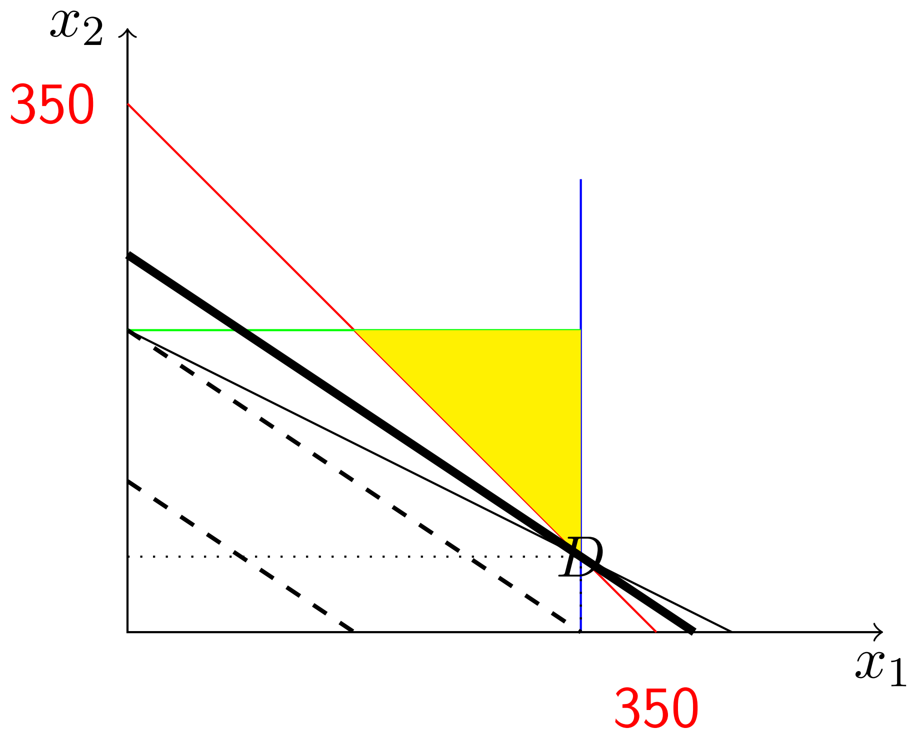

Optimal solution $D = (300, 50) \longrightarrow Z^* = 375$

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keeping at least 400 people employed instead of 300

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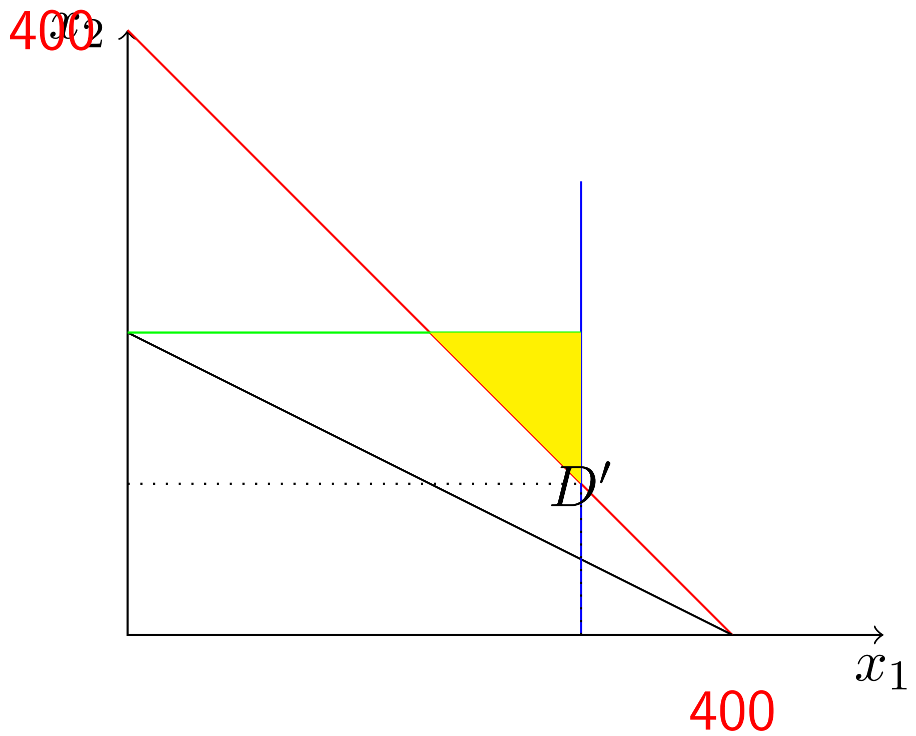
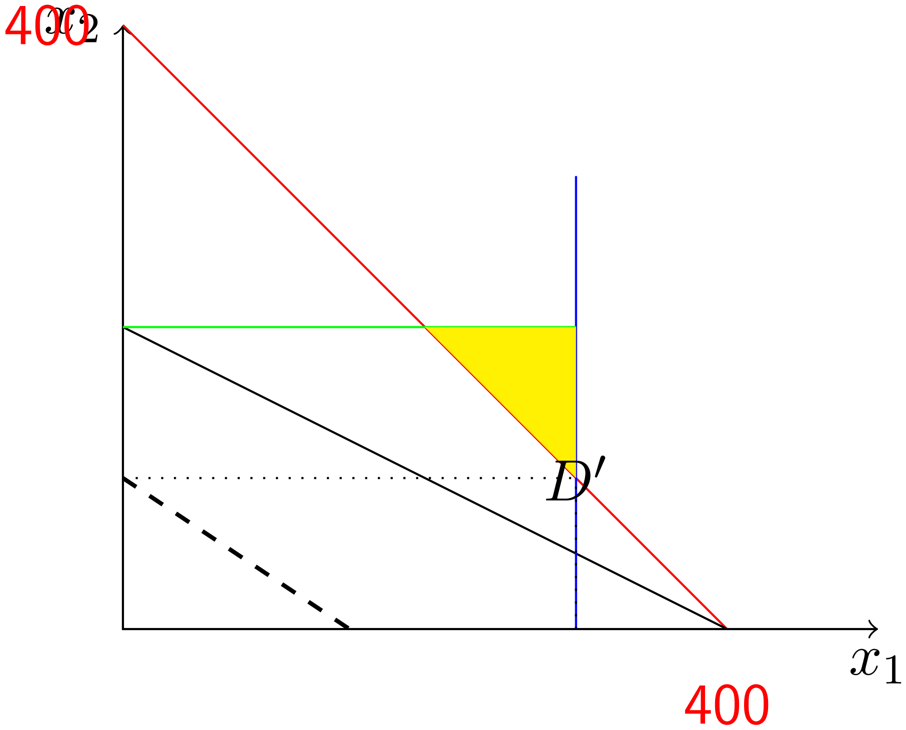


Optimal solution $D' = (300, 100) \longrightarrow Z^* = 450$

$$\frac{450 - 350}{100} = 1 \neq 0.5!$$

keeping at least 400 people employed instead of 300



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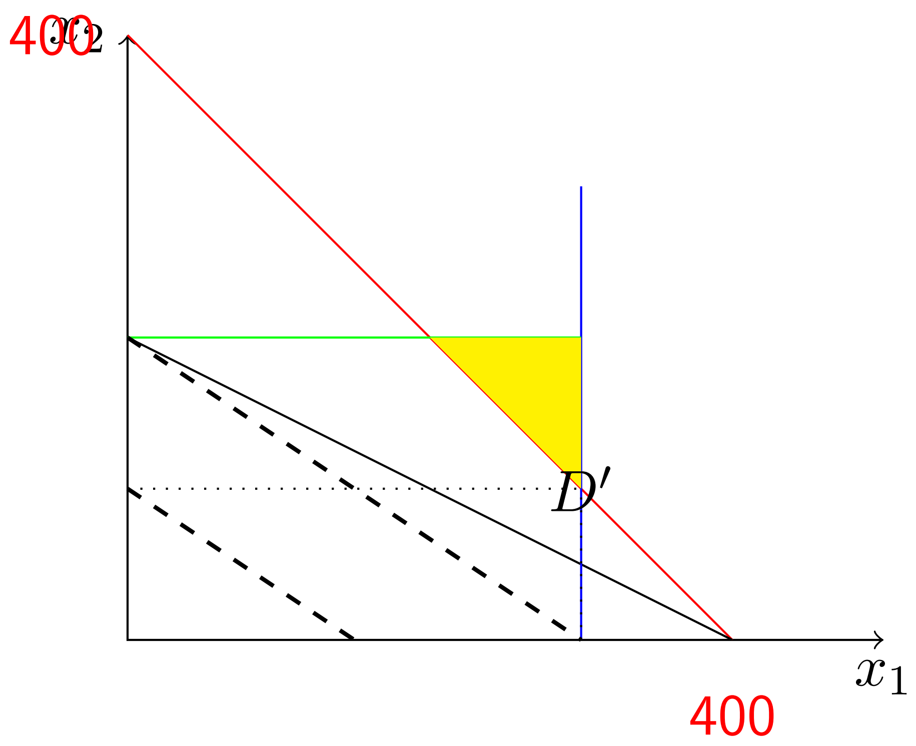

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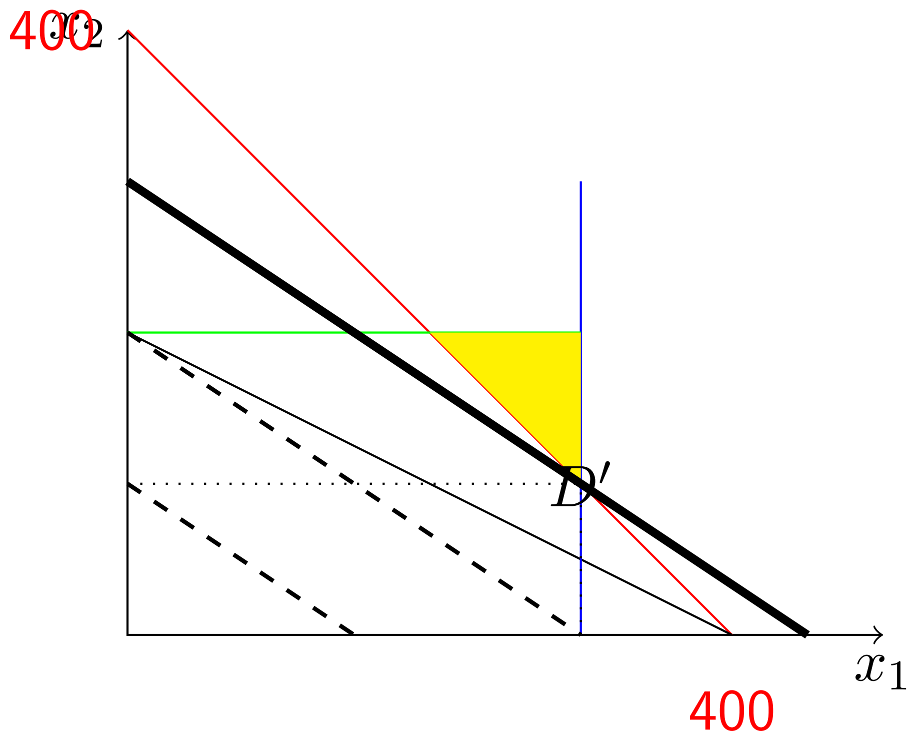

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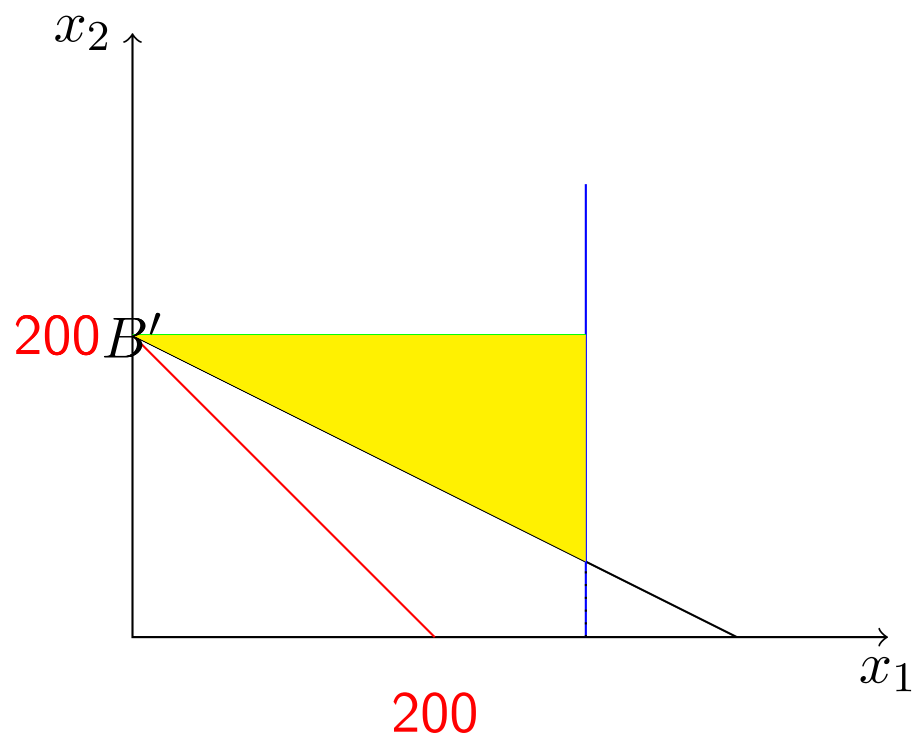
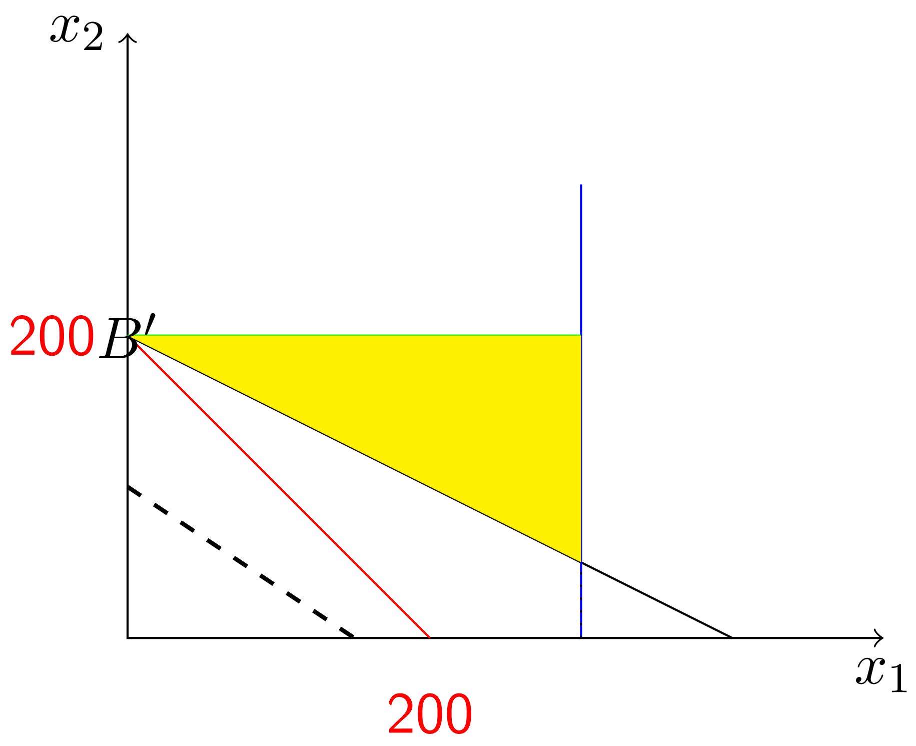


Optimal solution $B' = (0, 200) \longrightarrow Z^* = 300$

$$\frac{300 - 350}{100} = -0.5 \text{ units of BOD/d/worker}$$

keeping at least 200 people employed instead of 300

▷ Sensitivity analysis



Optimal solution $B' = (0, 200) \longrightarrow Z^* = 300$

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keeping at least 200 people employed instead of 300

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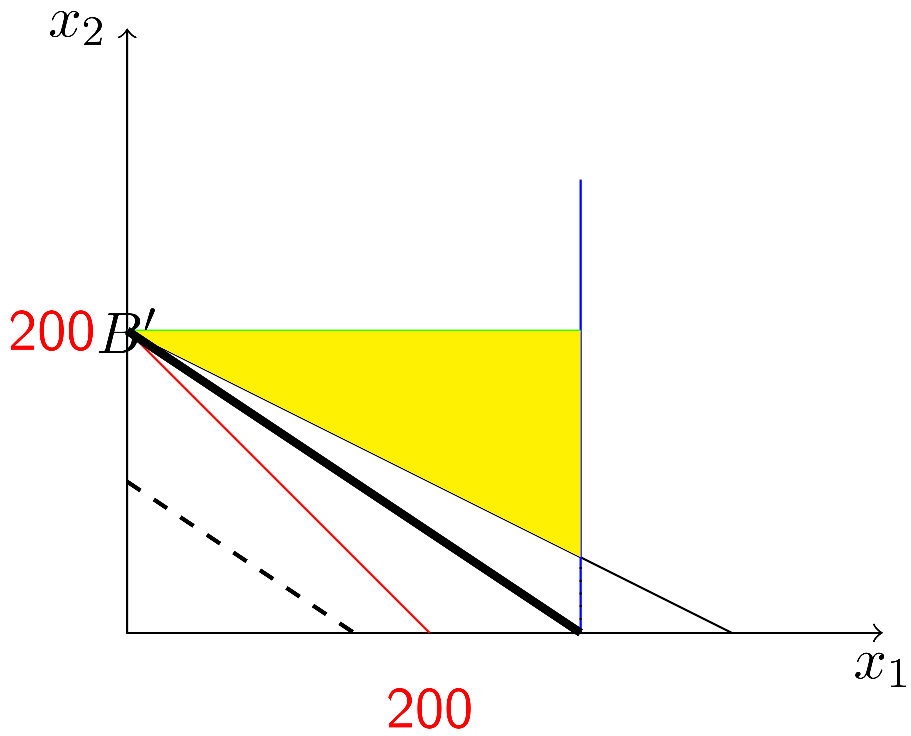
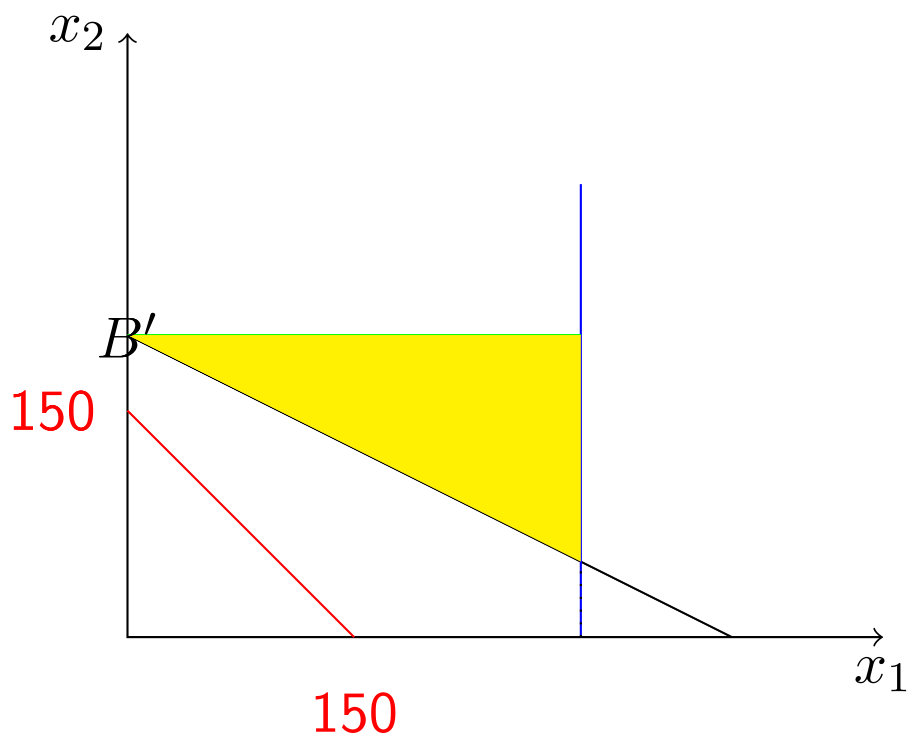


Optimal solution $B' = (0, 200) \longrightarrow Z^* = 300$

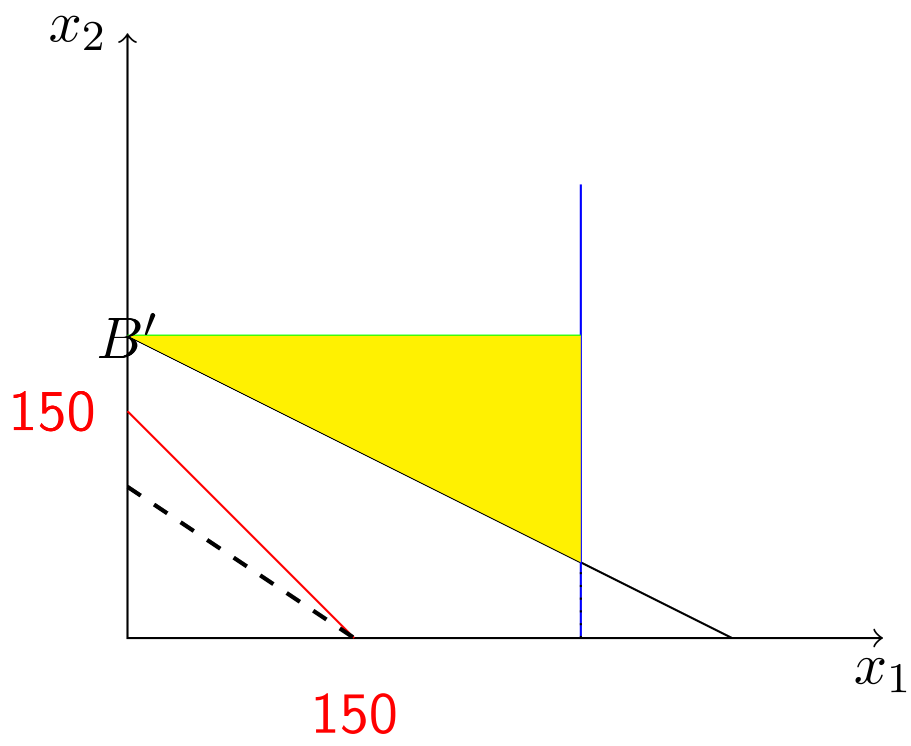
$$\frac{300 - 350}{100} = -0.5 \text{ units of BOD/d/worker}$$

keeping at least 150 people employed instead of 300



Optimal solution $B' = (0, 200) \longrightarrow Z^* = 300$

$$\frac{300 - 350}{150} = -0.3(3) \neq 0.5!$$



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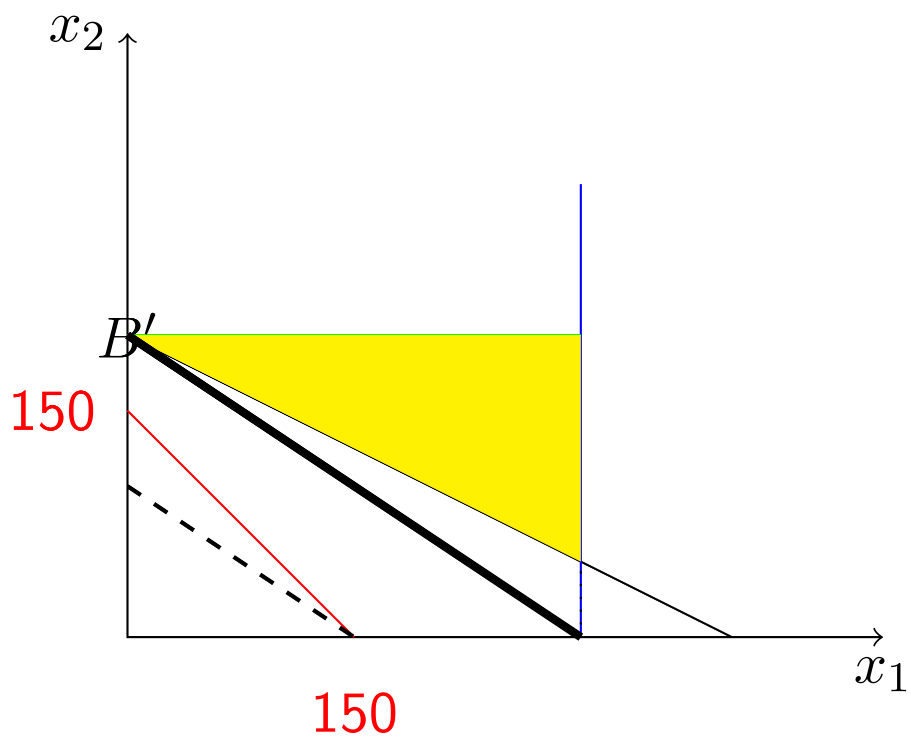
keeping at least 150 people employed instead of 300

▷ Sensitivity analysis



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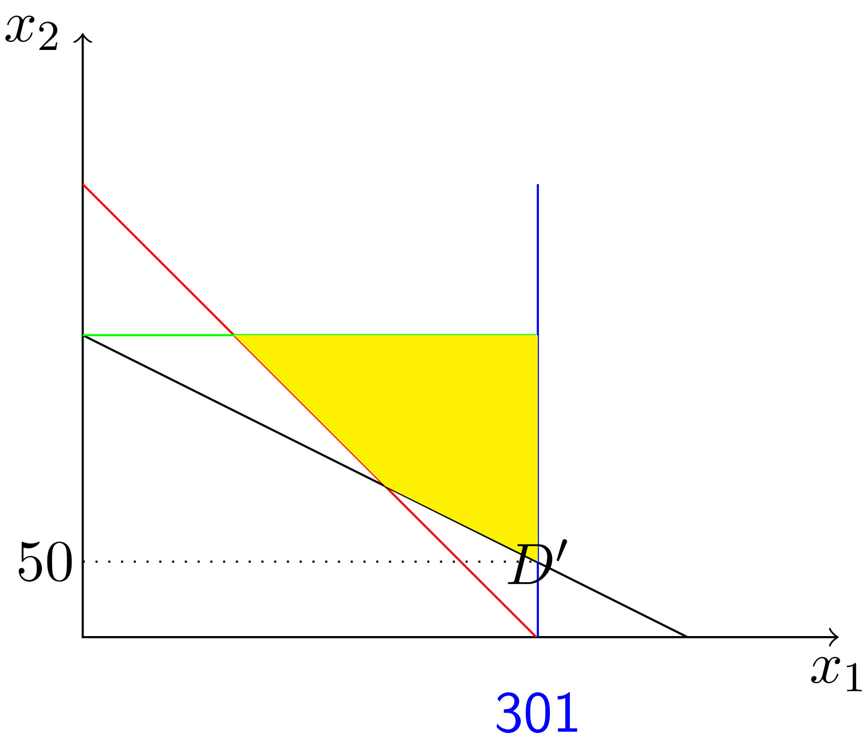
Increasing the mechanical pulp capacity from 300 to 301

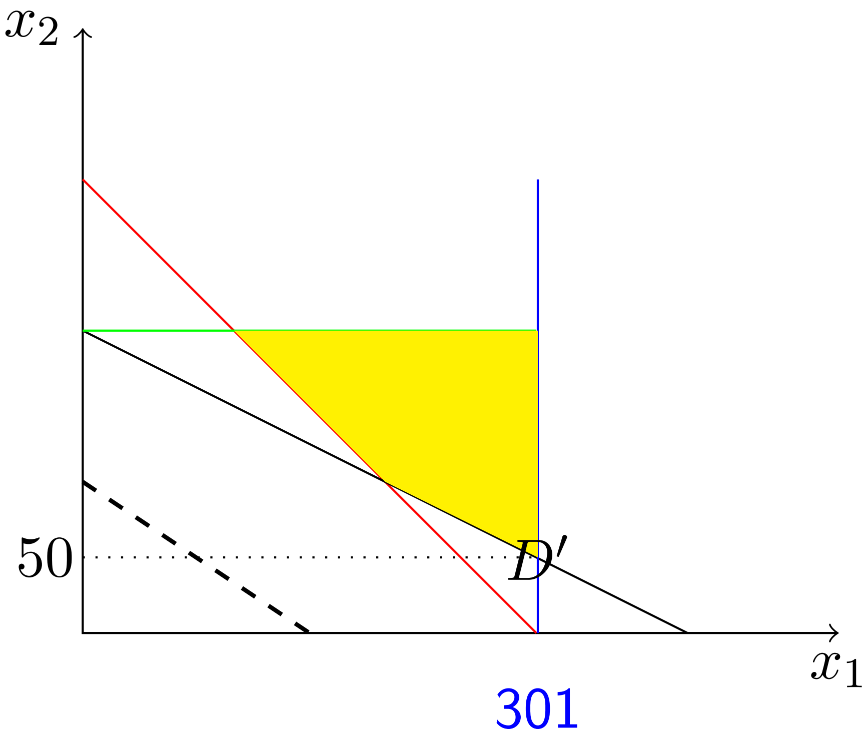


Optimal solution $A = (200, 100) \longrightarrow Z^* = 350$

$$\frac{350 - 350}{1} = 0 \text{ units of BOD/d/worker.}$$

Increasing the mechanical pulp capacity from 300 to 301

▷ Sensitivity analysis



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Increasing the mechanical pulp capacity from 300 to 301

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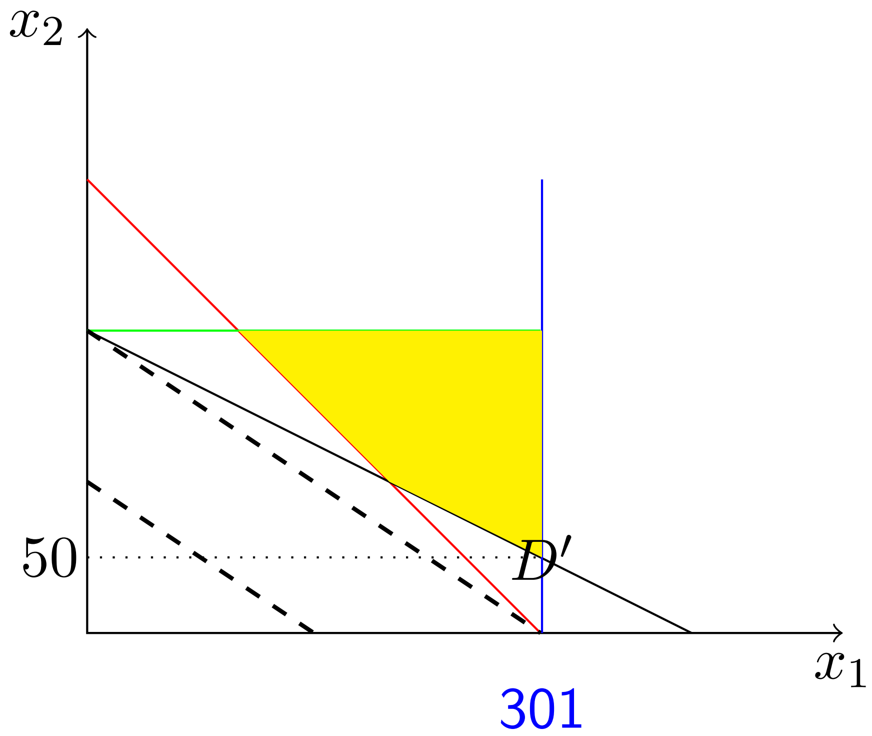
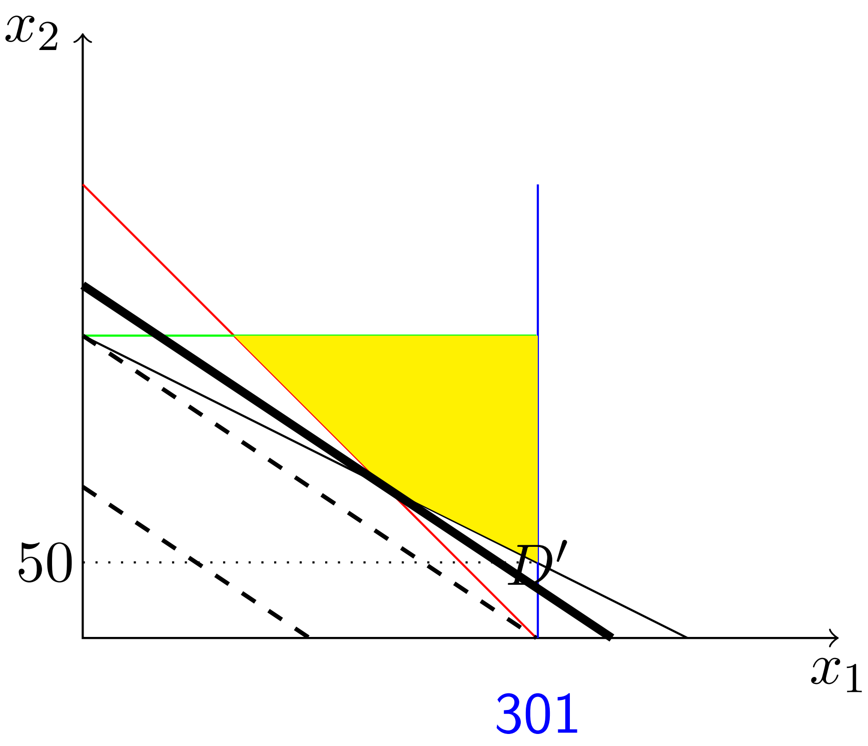


Optimal solution $A = (200, 100) \longrightarrow Z^* = 350$

$\frac{350 - 350}{1} = 0$ units of BOD/d/worker.

Increasing the mechanical pulp capacity from 300 to 301



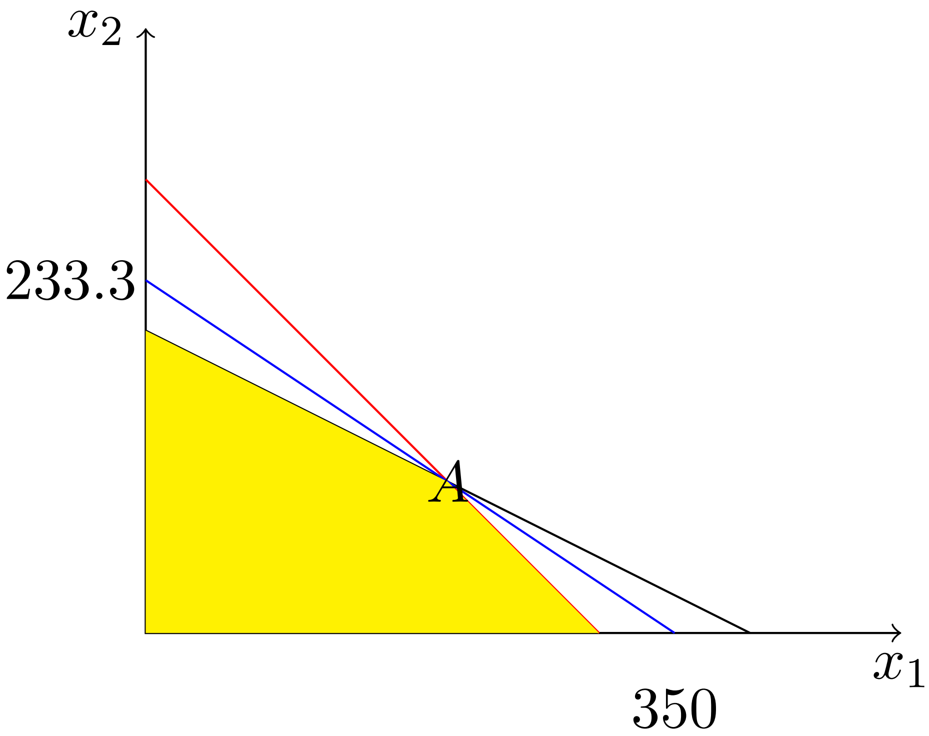
Optimal solution $A = (200, 100) \longrightarrow Z^* = 350$

$$\frac{350 - 350}{1} = 0 \text{ units of BOD/d/worker.}$$

Binding constraint - constraint that is satisfied in the equality by the optimal solution (constraints (2) and (3) in the keeping the river clean problem)

The shadow price of a non-binding constraint is equal to zero!

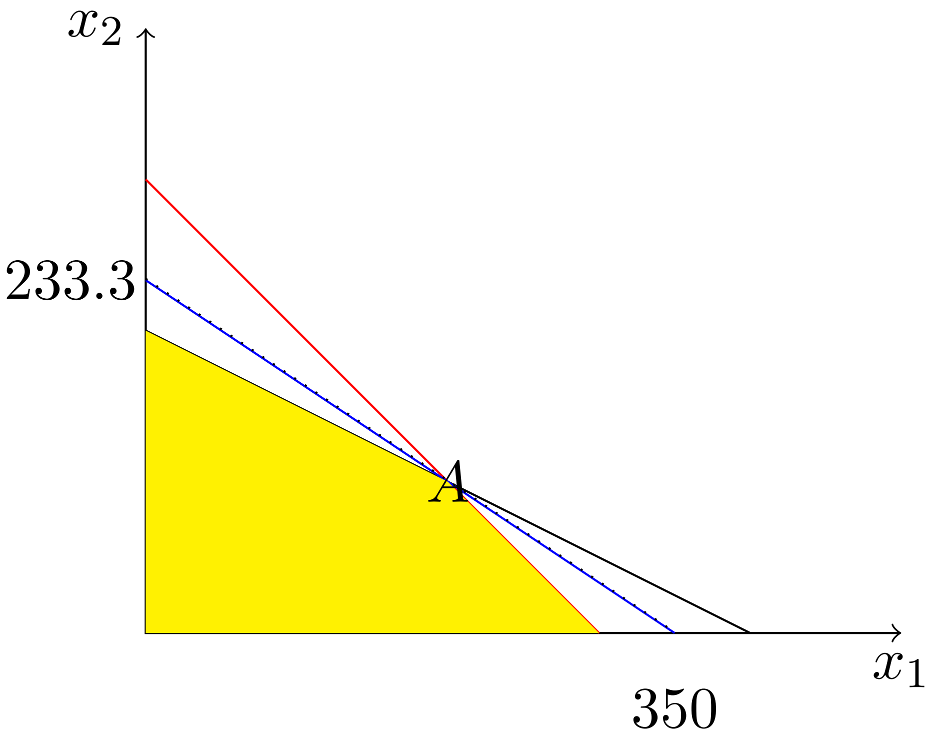
But, a binding constraint can have a shadow price equal to zero! How?



Optimal solution $A = (200, 100)$

Moving up the blue line will not change the optimal solution!

Right-hand-side values



Optimal solution $A = (200, 100)$

Moving up the blue line will not change the optimal solution!