# Integer Linear Programming

2018/19

▷ IP model

$$
\begin{array}{llll}
\text{Max} & Z = 4x_1 + 6x_2 & & (1) \\
\text{s.t.} & 0.5x_1 & \leq 2 & (2) \\
 & 2x_1 + 20x_2 & \leq 70 & (3) \\
 & 97.5x_1 + 136.5x_2 & \leq 682.5 & (4) \\
 & x_1, \quad x_2 & \text{are integers} & (5) \\
 & x_1, \quad x_2 & \geq 0 & (6)
\end{array}
$$

# Feasible region - set of white points

▷ IP model


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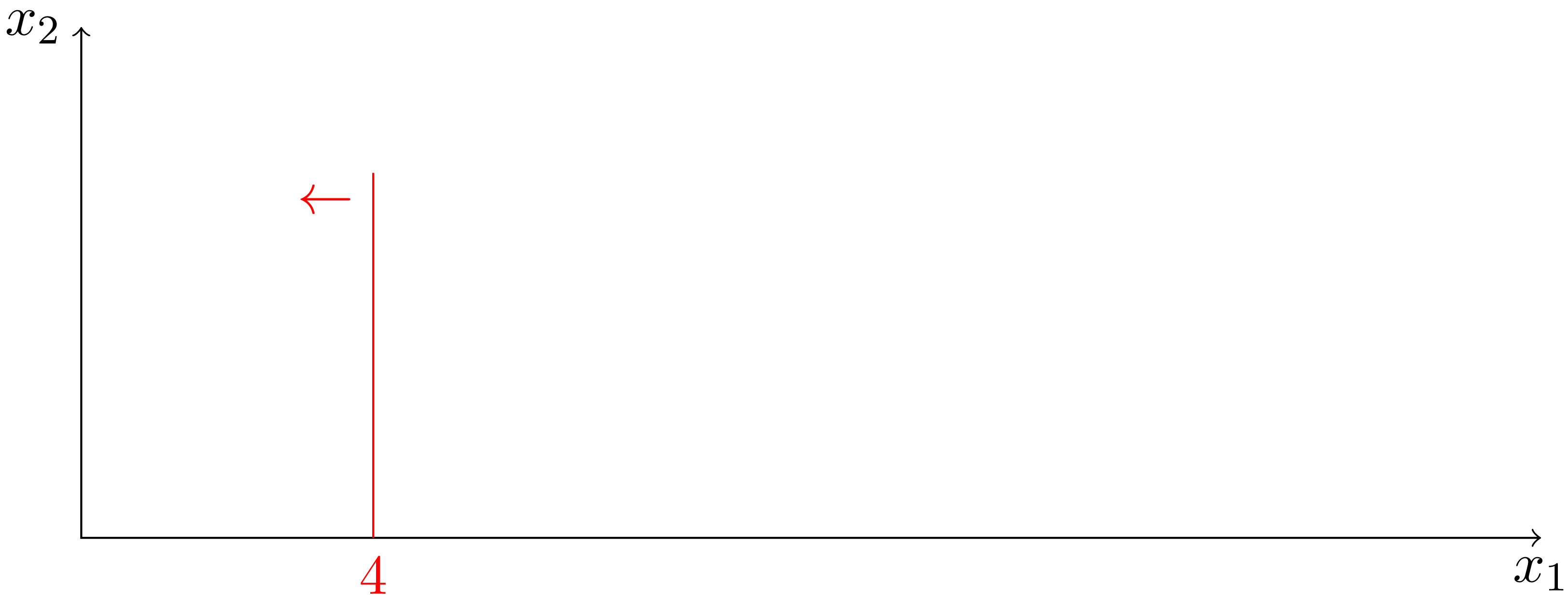
▷ IP model

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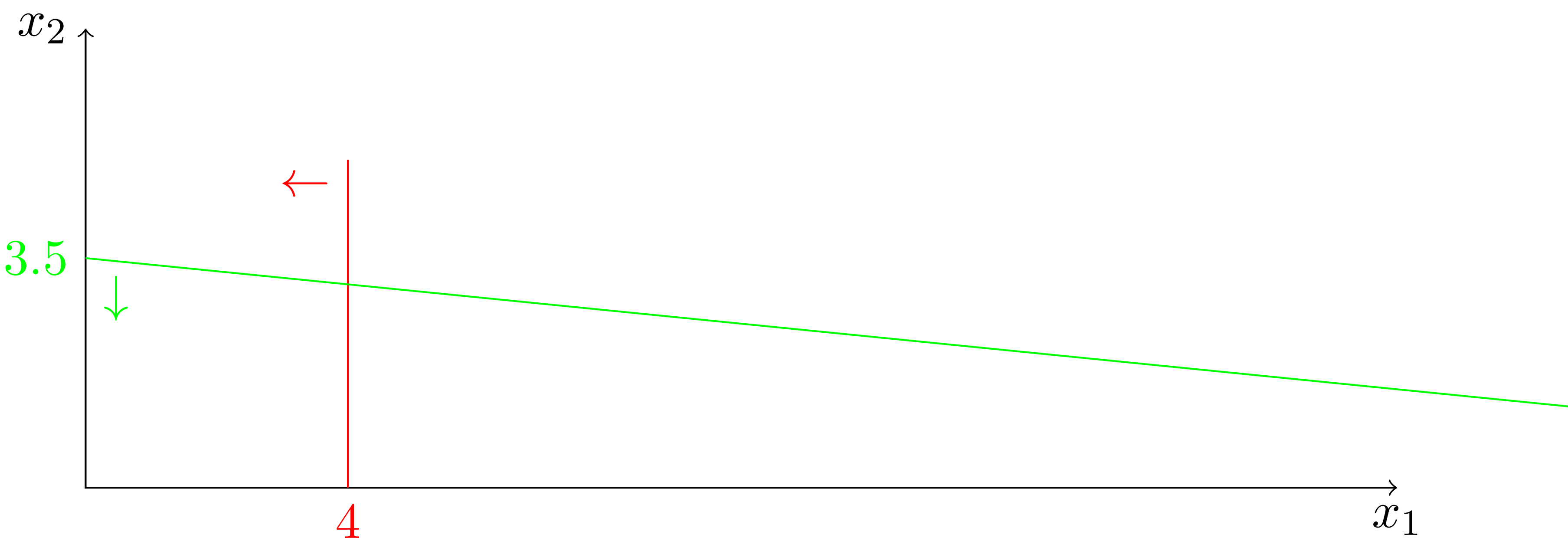


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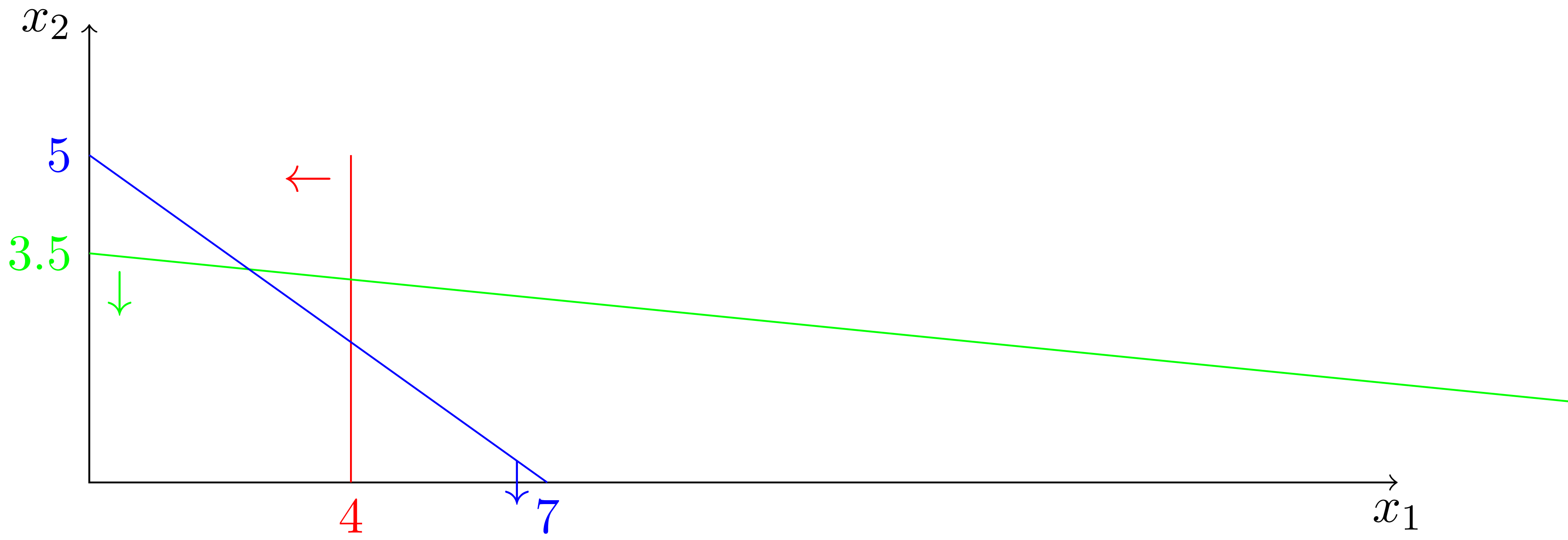


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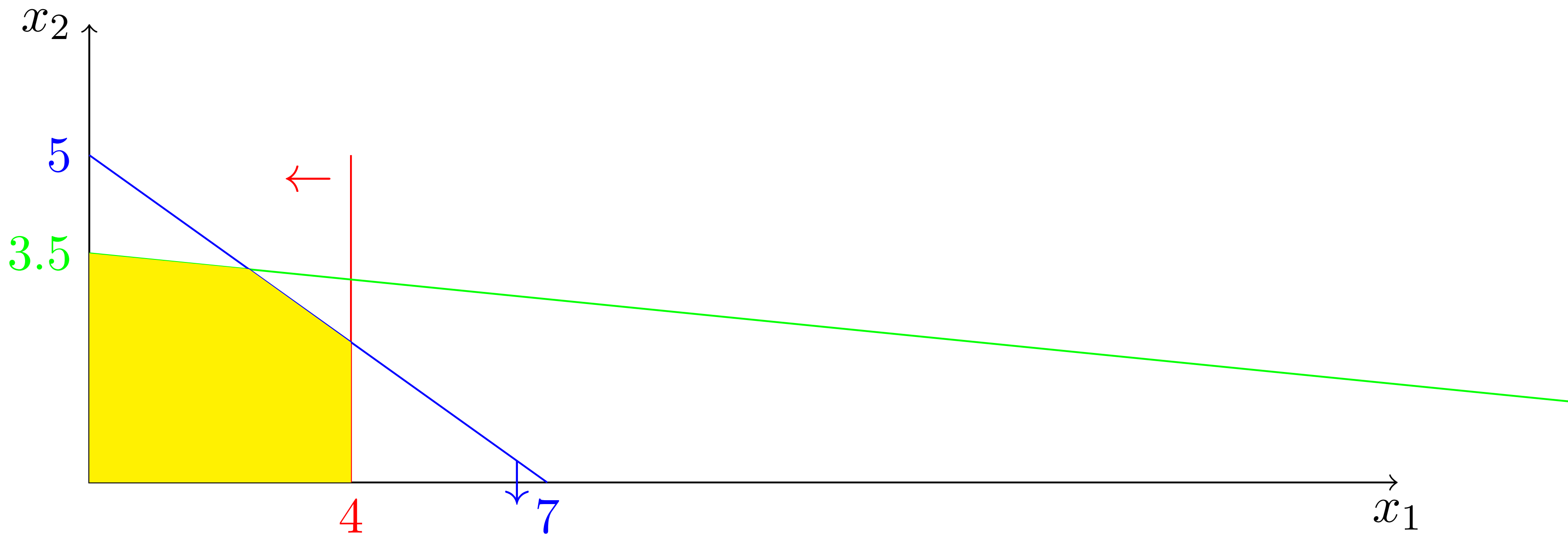


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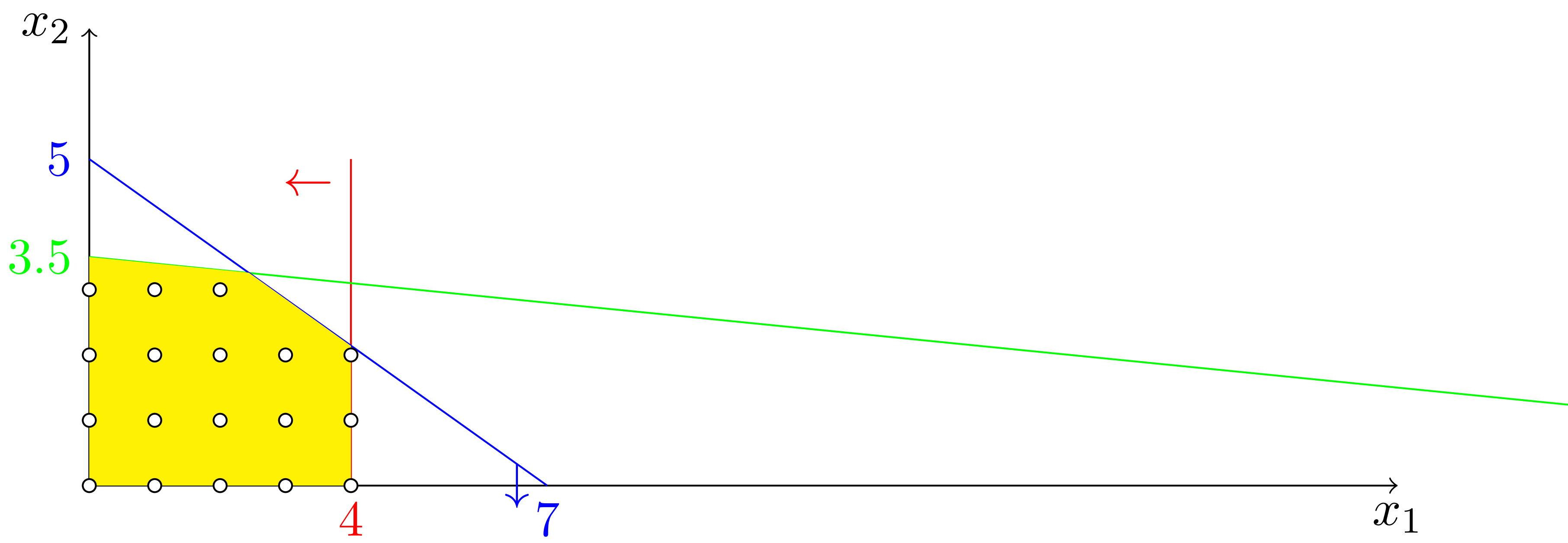


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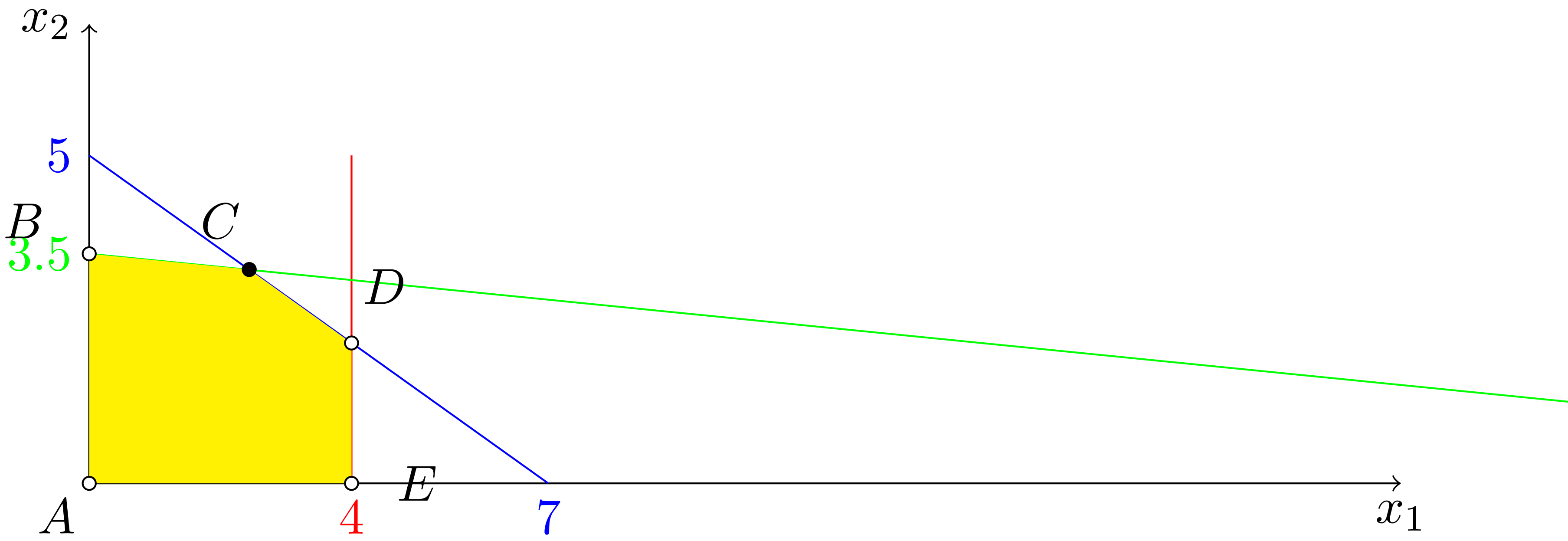
# Relaxing constraints (5) - linear relaxation

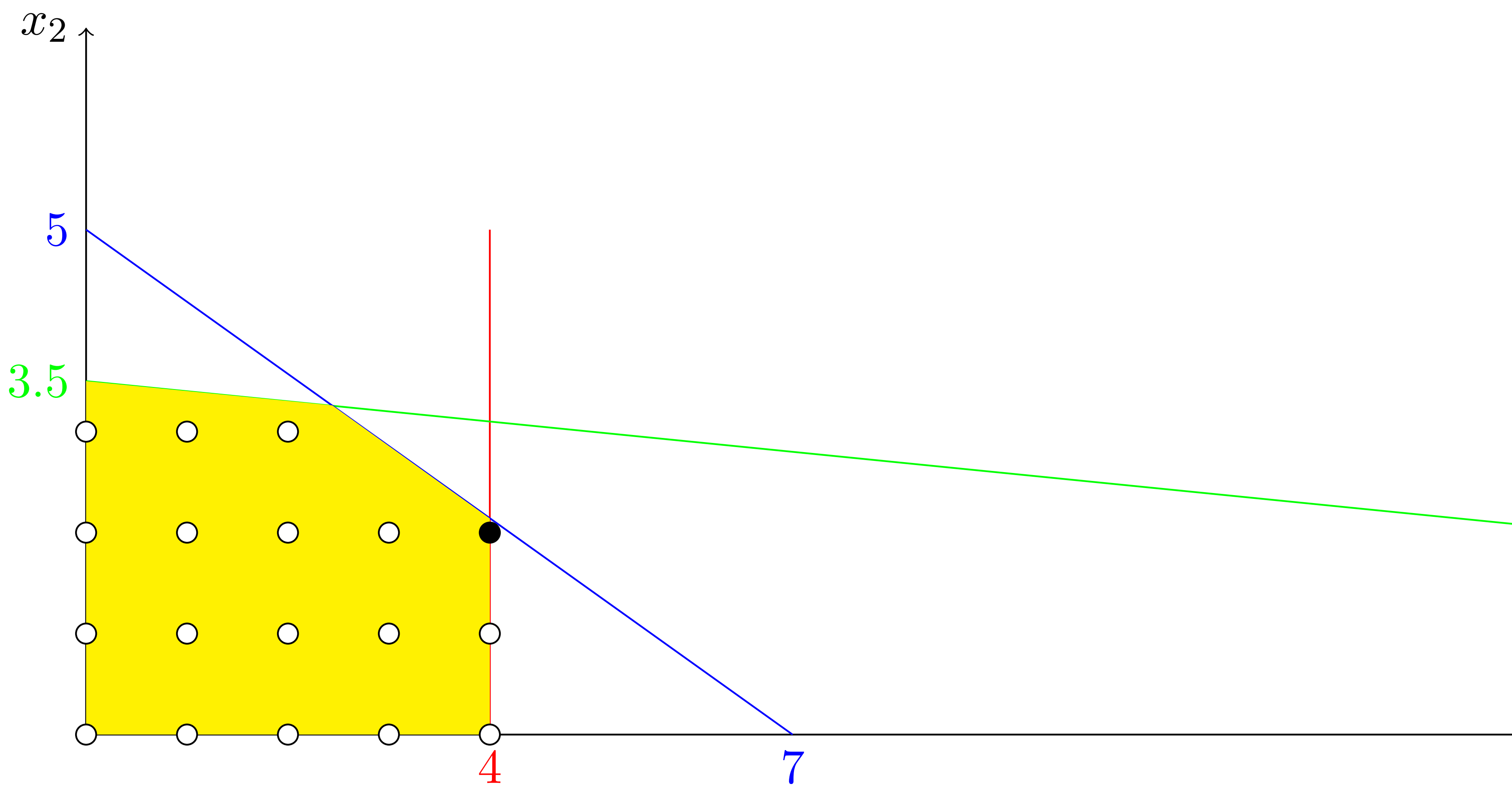
▷ IP model




| Vertex | $Z = 4x_1 + 6x_2$ |
|---|---|
| $A = (0, 0)$ | 0 |
| $B = (0, 3.5)$ | 14 |
| $C = (2.44, 3.26)$ | 29.3 (optimal solution) |
| $D = (4, 2.14)$ | 28.8 |
| $E = (4, 0)$ | 24 |

# Coming back to the IP model

▷ IP model




Optimal solution $x_1 = 4$, $x_2 = 2$ with the objective value of 28 ((4,2) is not a vertex of the linear relaxation).

# Constraints Hiring rangers

▷ IP model

$$x_j = \begin{cases} 1 & \text{if a ranger is placed in district } j \\ 0 & \text{otherwise} \end{cases}$$

$x_1 + x_2 + x_3$ is the number of rangers that protect district 1

# Constraints Sheet cutting planning



$x_j$ = number of 48 cm x 96 cm sheets assigned to cutting pattern $P_j$

$2x_1$ is the number of sheets of type 2 obtained with cutting pattern $P_1$

# Constraints Project selection

▷ IP model

$$x_j = \begin{cases} 1 & \text{if project } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

If project 1 is selected then project 6 is selected

$$x_1 \leq x_6$$

$$x_1 = 1 \Rightarrow x_6 \geq 1 \quad \overbrace{\Rightarrow}^{x_6 \text{ is binary}} \quad x_6 = 1$$

$$x_1 = 0 \Rightarrow x_6 \geq 0 \Rightarrow x_6 = 1 \text{ or } x_6 = 0$$

If project 6 is selected then project 1 is selected

$x_1 \leq x_6$ does not guarantee this!

$$x_6 = 1 \Rightarrow x_1 \leq 1 \Rightarrow x_1 = 1 \text{ or } x_1 = 0$$

# Constraints Project selection

▷ IP model

If project 6 is selected then project 1 is selected

$$x_6 \leq x_1$$

$$x_6 = 1 \Rightarrow x_1 \geq 1 \Rightarrow x_1 = 1$$

$$x_6 = 0 \Rightarrow x_1 \geq 0 \Rightarrow x_1 = 1 \text{ or } x_1 = 0$$

Project 1 is selected if and only if project 6 is selected.

$$x_6 \leq x_1 \text{ and } x_6 \geq x_1 \Leftrightarrow x_6 = x_1$$

# Constraints Project selection



$$x_j = \begin{cases} 1 & \text{if project } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

If project 2 is selected then projects 4 and 5 must both be selected

$$x_2 \leq x_4$$

$$x_2 \leq x_5$$

$$x_2 = 1 \Rightarrow x_4 = 1$$
$$x_2 = 1 \Rightarrow x_5 = 1$$
$$x_2 = 0 \Rightarrow x_4 = 0 \text{ or } x_4 = 1$$
$$x_2 = 0 \Rightarrow x_5 = 0 \text{ or } x_5 = 1$$

# Constraints Project selection



$$x_j = \begin{cases} 1 & \text{if project } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

If projects 1 and 2 are both selected then 6 must be selected

$$x_1 + x_2 - 1 \leq x_6$$

$$x_1 = x_2 = 1 \Rightarrow x_6 = 1$$

$$x_1 = 0 \text{ and } x_2 = 1 \Rightarrow x_6 = 0 \text{ or } x_6 = 1$$

$$x_1 = 1 \text{ and } x_2 = 0 \Rightarrow x_6 = 0 \text{ or } x_6 = 1$$

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