Applied Operations Research

Solving applications of Integer Linear Programming with Excel (One resolution)

- 2018/2019-

1. **Hiring rangers**

(a) **Indicate the districts that a ranger can protect.**

Districts	Districts where the rangers can be placed										
	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	0	0	0	0	0	0	0	0
2	1	1	1	0	1	1	0	0	0	0	0
3	1	1	1	1	0	1	1	0	0	0	1
4	0	0	1	1	0	0	1	1	0	0	0
5	0	1	0	0	1	1	0	0	1	0	0
6	0	1	1	0	1	1	1	0	1	0	1
7	0	0	1	1	0	1	1	1	0	0	1
8	0	0	0	1	0	0	1	1	0	1	1
9	0	0	0	0	1	1	0	0	1	1	1
10	0	0	0	0	0	0	0	1	1	1	1
11	0	0	1	0	0	1	1	1	1	1	1

Table 1: Districts that a ranger can protect.

(b) **Formulate and solve the problem.**

The decision variables are as follows:

$$x_j = \begin{cases} 1 & \text{if a ranger is placed in district } j \\ 0 & \text{otherwise.} \end{cases}$$

The model is the following:

$$\min Z = \sum_{j=1}^{11} x_j \qquad (1)$$

subject to

$$
\begin{array}{lllllllllllll}
x_1 & +x_2 & +x_3 & & & & & & & & & \geq 1 \\
x_1 & +x_2 & +x_3 & & +x_5 & +x_6 & & & & & & \geq 1 \\
x_1 & +x_2 & +x_3 & +x_4 & & +x_6 & +x_7 & & & & +x_{11} & \geq 1 \\
 & & x_3 & +x_4 & & & +x_7 & +x_8 & & & & \geq 1 \\
 & x_2 & & & +x_5 & +x_6 & & & +x_9 & & & \geq 1 \\
 & x_2 & +x_3 & & +x_5 & +x_6 & +x_7 & & +x_9 & & +x_{11} & \geq 1 \\
 & & x_3 & +x_4 & & +x_6 & +x_7 & +x_8 & & & +x_{11} & \geq 1 \\
 & & & x_4 & & & +x_7 & +x_8 & & +x_{10} & +x_{11} & \geq 1 \\
 & & & & x_5 & +x_6 & & & +x_9 & +x_{10} & +x_{11} & \geq 1 \\
 & & & & & & & x_8 & +x_9 & +x_{10} & +x_{11} & \geq 1 \\
 & & x_3 & & & +x_6 & +x_7+ & x_8+ & x_9+ & x_{10}+ & x_{11} & \geq 1 \\
x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8, & x_9, & x_{10}, & x_{11} & \in \{0,1\}.
\end{array}
$$

Expression (1) minimizes the number of rangers hired. All constraints bebore the last ensure that each district is protected. The last constraints state the nature of the variables.

The optimal solution is obtained with three rangers, placed in districts 2, 3 and 11.

2. **Sheet cutting planning**

Formulate this problem as an IP model and solve the model.

Sheet of paper	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	0	2	1	0	3	2	1	0	5	4	3	2	1	0
3	0	0	0	2	2	2	1	1	1	1	0	0	0	0	0	0
4	0	1	3	0	1	3	0	2	3	5	0	1	3	5	6	8

Table 2: Possible cutting patterns on a 48 cm × 96 cm sheet.

The decision variables are as follows:

x_j- number of 48 cm $\times$ 96 cm sheets assigned to cutting pattern P_j, $j = 1, ..., 16$.

$$\min\ Z = \sum_{i=1}^{16} x_i$$

subject to

$$\begin{array}{llllllllllll}
x_1+ & x_2+x_3 & & & & & & & & & & = 800 \\
2x_1+ & x_2+ & 2x_4+x_5+ & & 3x_7+ & 2x_8+x_9+ & & 5x_{11}+ & 4x_{12}+3x_{13}+ & 2x_{14}+x_{15} & & = 1300 \\
 & & 2x_4+2x_5+ & 2x_6+x_7 & & x_8+x_9+ & x_{10} & & & & & = 500 \\
 & x_2+3x_3+ & x_5+ & 3x_6+ & & 2x_8+3x_9+ & 5x_{10}+ & & x_{12}+3x_{13}+ & 5x_{14}+6x_{15}+ & 8x_{16} & = 1500 \\
x_j & & & & & & & & & & & \in \mathbb{N}_0,\ i = 1, ..., 16.
\end{array}$$

The first expression minimizes the number of 48 cm × 96 cm sheets used. All constraints before the last ensure that the characteristics of the order are satisfied. The last constraints state the integer requirements on the variables.

The optimal number of 48 cm × 96 cm sheets used is 1088. The optimal values of the variables are displayed in Table 3.

Number of 48 cm × 96 cm sheets															
P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}
642	4	154	1	0	249	0	0	0	0	1	0	0	1	3	33

Table 3: Optimal solution.

3. **Project selection**

   (a) **What is the optimal solution for this problem?**

   $x_1 = x_2 = x_3 = x_5 = 1$ and the remaining variables are equal to zero, with the objective function value of 22.

   (b) **Formulate constraints and find the optimal solutions for the following conditions:**

   i. **Exactly two projects out of projects 2, 3, 4 and 5 must be selected.**

$$x_2 + x_3 + x_4 + x_5 = 2.$$

   The optimal solution is $x_2 = x_3 = 1$ and $x_1 = x_4 = x_5 = x_6 = 0$, with the objective function value of 21.

   ii. **Project 1 may be selected if and only if project 6 is selected.**

$$x_1 = x_6.$$

   $x_2 = x_3 = x_4 = x_5 = 1$ and $x_1 = x_6 = 0$, with the objective function value of 21.

   iii. **If project 2 is selected, projects 4 and 5 must both be selected.**

$$x_2 \leq x_4$$

$$x_2 \leq x_5$$

$x_2 = x_3 = x_4 = x_5 = 1$ and $x_1 = x_6 = 0$, with the objective function value of 21.

iv. **If project 1 and 2 are both selected, 6 must be selected.**

$$x_1 + x_2 - 1 \leq x_6$$

$x_2 = x_3 = x_4 = x_5 = 1$ and $x_1 = x_6 = 0$, with the objective function value of 21.

4. **Road network design**

(a) **Formulate this problem as an IP model.**

The decision variables are as follows:

$$\text{For } i = 1, ..., 4,\ x_i = \begin{cases} 1 & \text{if project } P_i \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{For } j = 1, ..., 5,\ y_j = \begin{cases} 1 & \text{if road } j \text{ is built} \\ 0 & \text{otherwise.} \end{cases}$$

The model is as follows:

$$\min\ Z = 0.7x_1 + 0.1x_2 + 0.5x_3 + 0.8x_4 + 0.8y_1 + +0.4y_2 + 0.3y_3 + 0.2y_4 + 0.4y_5 \quad (1)$$

subject to

$$x_1 + x_2 + 2x_3 + 3x_4 \geq 2 \quad (2)$$

$$6x_1 + 8x_2 + 13x_3 + 10x_4 \geq 17 \quad (3)$$

$$x_1 \leq y_1 \quad (4)$$

$$x_2 = y_2 \quad (5)$$

$$x_2 \leq y_1 \quad (6)$$

$$x_3 = y_4 \quad (7)$$

$$x_3 \leq y_3 \quad (8)$$

$$x_3 \leq y_1 \quad (9)$$

$$x_4 = y_5 \quad (10)$$

$$x_4 \leq y_3 \quad (11)$$

$$x_4 \leq y_1 \quad (12)$$

$$x_i \in \{0, 1\},\ \ i = 1, ..., 4 \quad (13)$$

$$y_j \in \{0, 1\},\ \ j = 1, ..., 5. \quad (14)$$

Expression (1) minimizes the cost. Constraint (2) ensures the minimum number of recreational visitor days required. Constraint (3) ensures the minimum value for timber production required. Constraints (4) to (12) guarantee the road construction whenever necessary. Constraints (13) and (14) state the binary requirement on the variables.

Other alternative model is as follows:

$$\min\ Z = 0.7x_1 + 0.1x_2 + 0.5x_3 + 0.8x_4 + 0.8y_1 + +0.4y_2 + 0.3y_3 + 0.2y_4 + 0.4y_5 \tag{15}$$

$$subject\,to$$

$$x_1 + x_2 + 2x_3 + 3x_4 \geq 2 \tag{16}$$

$$6x_1 + 8x_2 + 13x_3 + 10x_4 \geq 17 \tag{17}$$

$$x_1 \leq y_1 \tag{18}$$

$$x_2 = y_2 \tag{19}$$

$$x_3 = y_4 \tag{20}$$

$$x_4 = y_5 \tag{21}$$

$$y_4 \leq y_3 \tag{22}$$

$$y_5 \leq y_3 \tag{23}$$

$$y_2 \leq y_1 \tag{24}$$

$$y_3 \leq y_1 \tag{25}$$

$$x_i \in \{0, 1\},\ \ i = 1, ..., 4 \tag{26}$$

$$y_j \in \{0, 1\},\ \ j = 1, ..., 5. \tag{27}$$

Constraints (22) and (23) ensures that if road sections 4 or 5 are built then road section 3 is also built. Constraints (24) and (25) ensures that if road sections 2 or 3 are built then road section 1 is also built.

Constraints (26) and (27) define the nature of the variables.

(b) **Solve the model.**

The optimal annual cost is 2.3×10^6 €. The optimal solution found is in Table 4.

(c) **P_2 and P_3 project areas share a long common boundary. Because all projects involve building a road and timber harvesting, it may be desirable, for esthetic reasons, to avoid doing projects on adjacent land areas. Add this requirement to the model and solve the new model.**

Projects				Road sections				
x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5
0	1	1	0	1	1	1	1	0

Table 4: Optimal variable values.

$$x_2 + x_3 \leq 1 \tag{28}$$

Constraint (28) ensures that both projects P_2 and P_3 are not selected simultaneously. The optimal annual cost is 2.5×10^6 €, larger than that obtained in a). The optimal solution found is in Table 5.

Projects				Road sections				
x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5
1	0	1	0	1	0	1	1	0

Table 5: Optimal variable values.

(d) **Consider that the management objectives are to meet the timber production goal of 17000 m^3/y at least or the recreational goal of 2000 rvd/y at least (or both). Formulate and solve the new model.**

The new decision variables are as follows:

$$z_1 = \begin{cases} 1 & \text{if constraint } x_1 + x_2 + 2x_3 + 3x_4 \geq 2 \text{ is added to the model} \\ 0 & \text{otherwise} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if constraint } 6x_1 + 8x_2 + 13x_3 + 10x_4 \geq 17 \text{ is added to the model} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_1 + x_2 + 2x_3 + 3x_4 \geq 2z_1 \tag{29}$$

$$6x_1 + 8x_2 + 13x_3 + 10x_4 \geq 17z_2 \tag{30}$$

$$z_1 + z_2 \geq 1 \tag{31}$$

$$z_1, z_2 \in \{0, 1\}. \tag{32}$$

Constraints (29), (30), (31) and (32) ensure that one of constraints (16) and (17), at least, is satisfied. U_1 and U_2 can be equal to 2 and 17, respectively, or larger.

The optimal annual cost is 1.8×10^6 €, smaller than that obtained in a). The optimal solution found is in Table 6. Project P_3 satisfies constraint (16) but not constraint (17).

Projects				Road sections						
x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	z_1	z_2
0	0	1	0	1	0	1	1	0	1	0

Table 6: Optimal variable values.

(e) **Consider that the management objectives are to meet either the timber production goal of 17000 m^3/y at least or the recreational goal of 2000 rvd/y at least (but not both). Formulate and solve the new model.**

We have to replace constraint (31) from the previous model by constraint (33).

$$z_1 + z_2 = 1. \tag{33}$$

The optimal solution is the same as previously.

(f) **Consider that the management objectives are if timber production is greater than or equal to 17000 m^3/y, all camps taken together must be able to accommodate at least 2000 rvd/y. Consider also that one project at least should be selected. Formulate and solve the new model.**

We have to replace constraint (33) from the previous model by constraint (34).

$$z_2 \leq z_1 \tag{34}$$

and add constraint (35)

$$x_1 + x_2 + x_3 + x_4 \geq 1. \tag{35}$$

Constraint (34) ensures that if constraint (17) is added to the model ($z_2 = 1$) then constraint (16) is also added ($z_1 = 1$). Constraint (35) ensures that at least one project is selected.

The optimal annual cost is 1.3×10^6 €, smaller than that obtained in a). The optimal solution found is in Table 7. Project P_2 does not satisfy both constraints (16) and (17).

Projects				Road sections						
x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	z_1	z_2
0	1	0	0	1	1	0	0	0	0	0

Table 7: Optimal variable values.

(g) **Consider that P_3 can be partially realized (in this case, the project is done using a fraction of the land that could be allocated to the project). Assume that if the project is started, 0.3 million of euros is needed for basic infrastructure. Thereafter, costs increase in proportion to the level of completion. The annual productions of wood and recreation are relative to the project carried out in full, being directly proportional to the fraction of the project carried out. Formulate and solve the new model.**

We add the variable

$$w = \begin{cases} 1 & \text{if project } P_3 \text{ is selected} \\ 0 & \text{otherwise.} \end{cases}$$

and the changes in the formulation are as follows:

$$\min Z = 0.7x_1 + 0.1x_2 + 0.5x_3 + 0.8x_4 + 0.8y_1 + +0.4y_2 + 0.3y_3 + 0.2y_4 + 0.4y_5 + 0.3w \tag{36}$$

subject to

$$x_3 \leq w \tag{37}$$

$$x_i \in \{0,1\},\ i = 1,2,4 \tag{38}$$

$$0 \leq x_3 \leq 1 \tag{39}$$

$$y_j \in \{0,1\},\ j = 1,...,5 \tag{40}$$

$$w \in \{0,1\}. \tag{41}$$

The optimal annual cost is 2.45×10^6 €. The optimal solution found is in Table 8.

Projects				Road sections					
x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	w
0	1	0.69	0	1	1	1	1	0	1

Table 8: Optimal variable values.

5. **Route selection**

(a) **Compute the profit obtained with one ton of apples for each route.**

r	S_r	I_r	Non-fixed cost associated to operation i $(b_i + d_i)$									$P_r = S_r - [I_r + \sum_{i=1}^{9}(b_i + d_i) + 200]$
			1	2	3	4	5	6	7	8	9	
	(€/t)	(€/t)	(€/t)									(€/t)
1	1200	70	82	187	163	185						313
2	1100	80	82	187		185	183					183
3	1000	80	82					62	95	186	44	251
4	700	90	82					62	95		44	127
5	400	30									44	126
6	0	0										-200

Table 9: Profit obtained with one ton of apples for each route (P_r) (the annual equipment cost associated to each route is not considered).

(b) **Compute the annual equipment cost associated to each route.**

(c) **Find the optimal route selection so that the company can maximize profits.**

The decision variables are as follows, for r=1,...,6:

x_r - amount of apples distributed through route r (ton)

$$y_r = \begin{cases} 1 & \text{if route } r \text{ is selected} \\ 0 & \text{otherwise.} \end{cases}$$

The model is the following:

r	Fixed cost associated to operation i (a_i)									$E_r = \sum_{i=1}^{9} a_i$
	1	2	3	4	5	6	7	8	9	
	(€/year)									(€/year)
1	1300	7200	4200	7000						19700
2	1300	7200		7000	2000					17500
3	1300					600	1200	3600	700	7400
4	1300					600	1200		700	3800
5									700	700
6										0

Table 10: Annual equipment cost associated to each route (E_r).

$$\max Z = \sum_{r=1}^{6} (P_r x_r - E_r y_r) \tag{42}$$

subject to

$$\sum_{r=1}^{6} x_r = 1400 \tag{43}$$

$$L_r \leq x_r \leq U_r,\ r = 1, ..., 6 \tag{44}$$

$$60 \leq x_1 + x_2 + x_3 + x_4 \leq 1400 \tag{45}$$

$$300 \leq x_1 + x_2 \leq 1400 \tag{46}$$

$$300 \leq x_1 \leq 1400 \tag{47}$$

$$150 \leq x_1 + x_2 \leq 1400 \tag{48}$$

$$150 \leq x_2 \leq 1400 \tag{49}$$

$$100 \leq x_3 + x_4 \leq 1400 \tag{50}$$

$$150 \leq x_3 + x_4 \leq 1400 \tag{51}$$

$$120 \leq x_3 \leq 1400 \tag{52}$$

$$100 \leq x_3 + x_4 + x_5 \leq 1400 \tag{53}$$

$$x_r \leq U_r y_r,\ r = 1, ..., 6 \tag{54}$$

$$x_r \geq 0,\ r = 1, ..., 6 \tag{55}$$

$$y_r \in \{0, 1\},\ r = 1, ..., 6. \tag{56}$$

Expression (42) maximizes the profit. Constraint (43) ensures the distribution of 1400 ton of apples. Constraints (44) guarantee the capacity of the routes. Constraints (45) to (53) ensure the capacity of the operations. Constraints (54) impose that if apples are sold through a route then this route should be selected. Constraints (55) and (56) define the nature of the variables.

The optimal profit is 347170 € and the optimal variable values are in Table 11.

x's						y's					
x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	y_5	y_6
500	400	120	150	200	30	1	1	1	1	1	1

Table 11: Optimal variable values.