Goal Programming

An Analysis of Multiple-Objective Optimization

Susana Barreiro

8 May 2020

Goal Programming

- We have assumed so far that **linear programming** encompasses a **single overriding objective** (e.g. maximizing total profit / minimizing total cost).
- Most times this is not realistic since we frequently focus on a variety of objectives, e.g. forest management:
    - to maintain stable profit,
    - increase wood production,
    - diversify ecosystem services,
    - restrain the impact of pests /diseases,
    - minimize erosion,
    - …

Goal programming provides a way of achieving **several objectives *simultaneously*.**

Capacity limits we cannot change (e.g. number of seats on a flight) or we do not want to change

Goal Programming

- **Linear programming**

Most LP problems have **hard constraints** that cannot be violated:

Max: $Z = 90\,x_1 + 120\,x_2$

Subject to:

$x_1 \leq 40$ (ha of pine)

$x_2 \leq 50$ (ha of eucalypt)

$2x_1 + 3x_2 \leq 180$ (days of work)

and $x_1 \geq 0; \quad x_2 \geq 0$

Constraints are very important because they refer to the amount of resources / capacity limits we face

First, we look at our limitations;

Then, we think of an optimization model

Goal Programming

- **Linear programming**

  Most LP problems have **hard constraints** that cannot be violated:

  Max: $Z = 90\,x_1 + 120\,x_2$

  Subject to:

  $x_1 \leq 40$ (ha of pine)

  $x_2 \leq 50$ (ha of eucalypt)

  $2x_1 + 3x_2 \leq 180$ (days of work)

  and $x_1 \geq 0; \quad x_2 \geq 0$

- **Goal programming**

  GP problems have **soft constraints** that represent goals or targets we want to achieve

  Suppose we look back to the Poets problem again and he says that he reconsidered and would be:

  "… willing to give an extra 250 days of work if needed… preferred having 40 and 50 ha of pine and eucalypt but he would be flexible "

  The "days of work" would no longer be a hard constraint

Goal Programming

- The basic approach of **goal programming** is to:

  1) establish a specific numeric goal for each of the objectives
  2) formulate an objective function for each objective
  3) seek a solution that minimizes the (weighted) sum of deviations of these objective functions from their respective goals

Goal Programming

There are three possible types of goals:

- A **lower, one-sided goal** sets a *lower limit* that we do not want to fall under (but exceeding the limit is fine).

- An **upper, one-sided goal** sets an *upper limit* that we do not want to exceed (but falling under the limit is fine).

- A **two-sided goal** sets a *specific target* that we do not want to miss on either side.

Goal Programming

- Goal programming problems can be categorized according to the **type of mathematical programming model** that it fits except for having multiple goals instead of a single objective:
  - linear programming,
  - integer programming,
  - nonlinear programming,
  - etc

*In class, we ill only consider **linear goal programming**—those goal programming problems that fit linear programming, but I'll refer to it just as **goal programming***

Goal Programming

- Goal programming problems can also be categorized according to **how the goals compare in importance**:

- **nonpreemptive goal programming** – if all the goals are of *roughly* *comparable in importance*

- **preemptive goal programming** – if there is a *hierarchy of priority levels* for the goals, so that the goals of primary importance receive first priority attention, those of secondary importance receive second-priority attention, and so forth (if there are more than two priority levels).

Goal Programming - Nonpreemptive

The OR department of the DEWRIGHT COMPANY has been assigned the task of determining which mix of products should be produced.

Management wants primary consideration given to the following three goals:

(1) achieving a long-run profit (NPV) of at least $125 million from these products

(2) maintaining the current employment level of 4,000 employees,

(3) holding the capital investment to less than $55 million.

Goal Programming - Nonpreemptive

However, it probably will not be possible to attain all these goals simultaneously, priorities have been discussed leading to setting a *penalty weight*:

1) 5 for missing the profit goal (per $1 million under),
2) 2 for going over the employment goal (per 100 employees) and 4 for going under this same goal
3) 3 for exceeding the capital investment goal (per $1 million over)

Each new product’s contribution to profit, employment level, and capital investment level is *proportional* to the rate of production.

Goal Programming - Nonpreemptive

These contributions per unit rate of production are shown in the table along with the goals and penalty weights.

Factor	Unit Contribution Product: 1	2	3	Goal (Units)	Penalty Weight
Long-run profit	12	9	15	$\geq$ 125 (millions of dollars)	5
Employment level	5	3	4	= 40 (hundreds of employees)	2(+), 4(−)
Capital investment	5	7	8	$\leq$ 55 (millions of dollars)	3

Goal Programming - Nonpreemptive

Goal Programming Formulation: The Dewright Company problem includes all three possible types of goals:

- profit goal is a lower one-sided goal: $\mathbf{12x_1 + 9x_2 + 15x_3 \geq 125}$
- employment goal is a two-sided goal: $\mathbf{5x_1 + 3x_2 + 4x_3 = 40}$
- investment goal is an upper one-sided goal: $\mathbf{5x_1 + 7x_2 + 8x_3 \leq 55}$

Where x_1, x_2, x_3 are the decision variables representing the production rates of products 1, 2, and 3, respectively and $x_1, x_2, x_3 \geq 0$

Goal Programming - Nonpreemptive

Linear Programming Formulation: Transform **goals** into **constraints**

- Subject to:

$$\begin{aligned} 12x_1 + 9x_2 + 15x_3 &\geq 125 \\ 5x_1 + 3x_2 + 4x_3 &= 40 \\ 5x_1 + 7x_2 + 8x_3 &\leq 55 \end{aligned}$$

Maybe we can't satisfy all goals, but we want to capture how much we can satisfy (find the deviations)

- Objective function:

The objective then is to choose the values of x_1, x_2, and x_3 that minimize

Z = 5(amount under the long-run profit goal)
+ 2(amount over the employment level goal)
+ 4(amount under the employment level goal)
+ 3(amount over the capital investment goal)

where no penalties are incurred for being over the long-run profit goal or for being under the capital investment goal

Goal Programming - Nonpreemptive

Linear Programming Formulation:

To express this mathematically, we introduce some *auxiliary variables* y_1, y_2, and y_3, to represent the deviations defined as follows:

$y_1 = 12x_1 + 9x_2 + 15x_3 - 125$ (*long-run* ***profit*** *minus the target*)
$y_2 = 5x_1 + 3x_2 + 4x_3 - 40$ (***employment*** *level minus the target*)
$y_3 = 5x_1 + 7x_2 + 8x_3 - 55$ (*capital* ***investment*** *minus the target*)

Since each y_i can be either positive or negative, and replace each one by the difference of two nonnegative variables:

$y_1 = y_1^+ - y_1^-$, where $y_1^+ \geq 0$, $y_1^- \geq 0$
$y_2 = y_2^+ - y_2^-$, where $y_2^+ \geq 0$, $y_2^- \geq 0$
$y_3 = y_3^+ - y_3^-$, where $y_3^+ \geq 0$, $y_3^- \geq 0$

y_i^+ represents the positive part of y_i variable (positive deviation)
y_i^- represents the negative part of y_i variable (negative deviation)

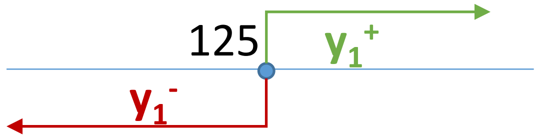
Goal Programming - Nonpreemptive

Linear Programming Formulation:

Subject to:

$$\boxed{12x_1 + 9x_2 + 15x_3 \geq 125}$$
$$5x_1 + 3x_2 + 4x_3 = 40$$
$$5x_1 + 7x_2 + 8x_3 \leq 55$$

If we're above 125, we have a positive deviation y_1^+, but it's according to the goal, so there is no problem. However, we don't want to have a negative deviation, **so we penalize y_1^-**



*N*ow we have a legitimate objective function for a linear programming model:

$$\textbf{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$$

Because there is no penalty for exceeding the profit goal of 125 or being under the investment goal of 55, neither y_1^+ nor y_3^- should appear in this objective function representing the total penalty for deviations from the goals.

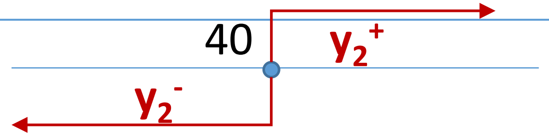
Goal Programming - Nonpreemptive

Linear Programming Formulation:

Subject to:

$$\begin{aligned} 12x_1 + 9x_2 + 15x_3 &\geq 125 \\ \boxed{5x_1 + 3x_2 + 4x_3 = 40} \\ 5x_1 + 7x_2 + 8x_3 &\leq 55 \end{aligned}$$

We don’t care if we meet exactly the goal of 40 (=40), but we don’t want to be above or below 40 (having positive or negative deviations, respectively), thus **we penalize y_2^+ both y_2^-**



*N*ow we have a legitimate objective function for a linear programming model:

$$\textbf{Min } Z = 5y_1^- + 2\,y_2^+ + 4\,y_2^- + 3\,y_3^+$$

Because there is no penalty for exceeding the profit goal of 125 or being under the investment goal of 55, neither y_1^+ nor y_3^- should appear in this objective function representing the total penalty for deviations from the goals.

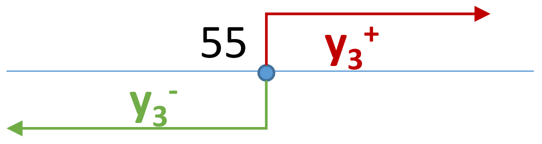
Goal Programming - Nonpreemptive

Linear Programming Formulation:

Subject to:

$$12x_1 + 9x_2 + 15x_3 \geq 125$$
$$5x_1 + 3x_2 + 4x_3 = 40$$
$$\boxed{5x_1 + 7x_2 + 8x_3 \leq 55}$$

We don’t care if we’re below 55, but we don’t want to be above (having positive deviation), thus **we penalize y_3^+**



*N*ow we have a legitimate objective function for a linear programming model:

$$\textbf{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$$

Because there is no penalty for exceeding the profit goal of 125 or being under the investment goal of 55, neither y_1^+ nor y_3^- should appear in this objective function representing the total penalty for deviations from the goals.

Goal Programming - Nonpreemptive

Linear Programming Formulation:

Finally, we must incorporate the above definitions of the y_i^+ and y_i^- directly into the model, because the simplex method considers only the objective function and constraints that constitute the model.

For example, since $y_1^+ - y_1^- = y_1$, the above expression for y_1 gives

$$12x_1 + 9x_2 + 15x_3 - 125 = y_1^+ - y_1^-$$

After we move the variables $(y_1^+ - y_1^-)$ to the left-hand side and the constant (125) to the right-hand side,

$$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$$

becomes a legitimate equality constraint for a linear programming model.

Goal Programming - Nonpreemptive

Linear Programming Formulation:

Proceeding in the same way for $y_2^+ - y_2^-$ and $y_3^+ - y_3^-$, we obtain the following formulation for this goal programming problem

- Objective function:

$$\textbf{Min } Z = 5y_1^- + 2\,y_2^+ + 4\,y_2^- + 3\,y_3^+$$

- Subject to:

$$\begin{aligned} 12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) &= 125 \\ 5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) &= 40 \\ 5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) &= 55 \end{aligned}$$

$$x_1\,,\, x_2\,,\, x_3\,,\, y_1^+,\, y_1^-,\, y_2^+,\, y_2^-\,,\, y_3^+\,,\, y_3^- \geq 0$$

Goal Programming - Preemptive

In the preceding example we assume that all the goals are of roughly comparable importance.

Now consider the case of *preemptive* goal programming, where there is a **hierarchy of priority levels for the goals**. Such a case arises when one or more of the goals clearly are far more important than the others.

Thus, the initial focus should be on achieving as closely as possible these *first-priority* goals, while the other goals might divide into *second-priority* goals, *third-priority* goals, and so on.

After we find an optimal solution with respect to the *first-priority* goals, we can break any ties for the optimal solution by considering the *second-priority* goals. Any ties that remain after this reoptimization can be broken by considering the *third-priority* goals, and so on.

Goal Programming - Nonpreemptive

Excel Solver:

Applying the simplex method to this formulation yields an optimal solution with:

$x_1 = 23/5$, $x_2 = 0$, $x_3 = 5/3$

$y_1^+ = 0$, $y_1^- = 0$, $y_2^+ = 23/5$, $y_2^- = 0$, $y_3^+ = 0$, $y_3^- = 0$

Therefore, $y_1 = 0$, $y_2 = 23/5$, and $y_3 = 0$, so the first and third goals are fully satisfied, but the employment level goal of 40 is exceeded by 8 1/3 (833 employees). The resulting penalty for deviating from the goals is $Z = 16\ 2/3$.

Goal Programming - Nonpreemptive

Excel Solver:

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

$$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$$

$$5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) = 40$$

$$5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) = 55$$

$$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \ge 0$$

$$\text{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$$

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	0	125
employment	5	3	4	0	0	-1	1	0	0	0	40
investment	5	7	8	0	0	0	0	-1	1	0	55
lower bounds:	0	0	0	0	0	0	0	0	0		
decision variables:	0	0	0	0	0	0	0	0	0		
Penalty weights:				0	5	2	4	3	0		

number of penalty points incurred by missing the goals

Min z= 0

Goal Programming - Nonpreemptive

Excel Solver:

Deviations below the goal

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

	Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$	profit	12	9	15	-1	1	0	0	0	0	0	125
$5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) = 40$	employment	5	3	4	0	0	-1	1	0	0	0	40
$5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) = 55$	investment	5	7	8	0	0	0	0	-1	1	0	55
$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$	lower bounds:	0	0	0	0	0	0	0	0	0		
	decision variables:	0	0	0	0	0	0	0	0	0		
$\text{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$	Penalty weights:				0	5	2	4	3	0		

number of penalty points incurred by missing the goals

Min z= 0

Goal Programming - Nonpreemptive

Deviations above the goal

Excel Solver:

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

$$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$$

$$5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) = 40$$

$$5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) = 55$$

$$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$$

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	0	125
employment	5	3	4	0	0	-1	1	0	0	0	40
investment	5	7	8	0	0	0	0	-1	1	0	55
lower bounds:	0	0	0	0	0	0	0	0	0		
decision variables:	0	0	0	0	0	0	0	0	0		
Penalty weights:				0	5	2	4	3	0		

number of penalty points incurred by missing the goals

Min $Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$

Min z= 0

Goal Programming - Nonpreemptive

Excel Solver:

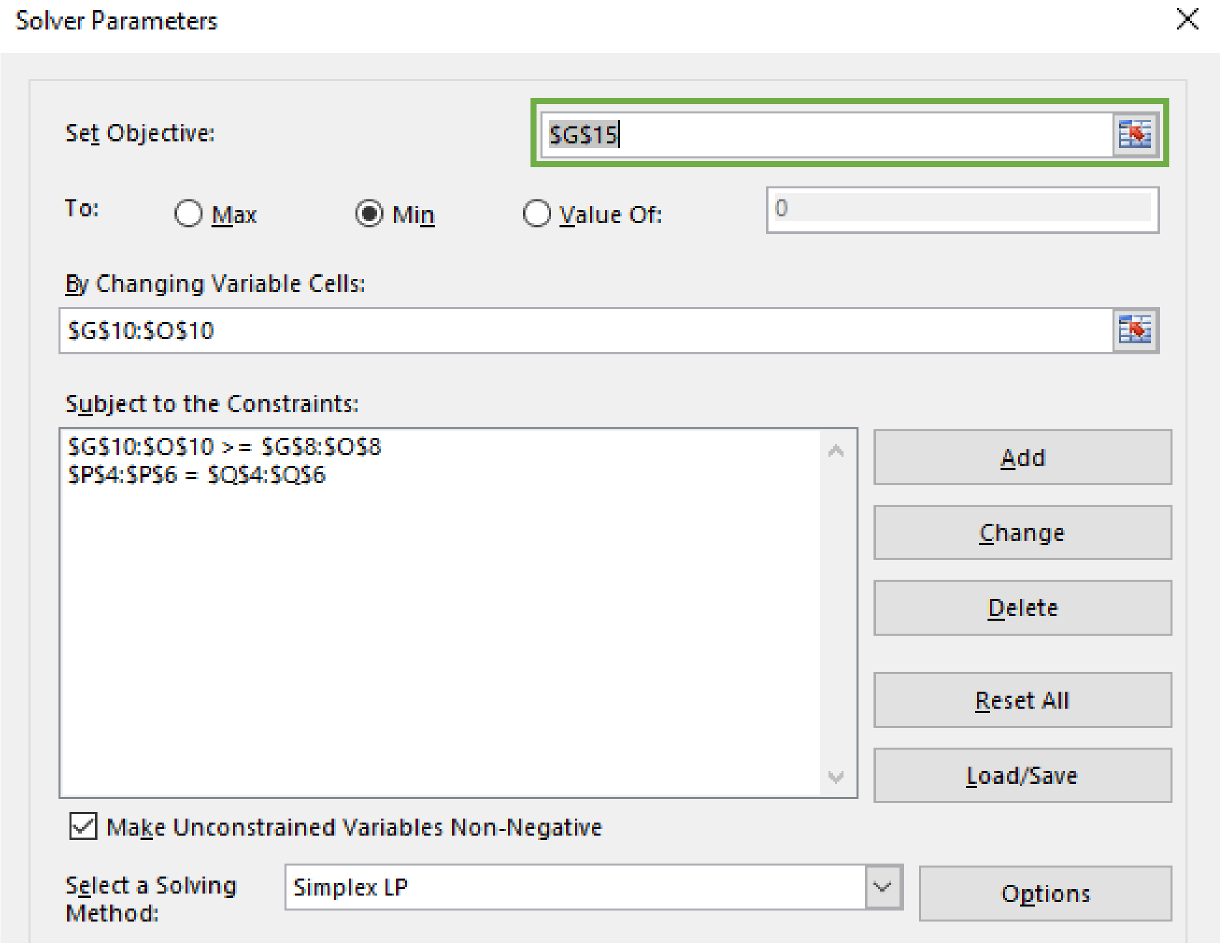
x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	0	125
employment	5	3	4	0	0	-1	1	0	0	0	40
investment	5	7	8	0	0	0	0	-1	1	0	55

	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-
lower bounds:	0	0	0	0	0	0	0	0	0
decision variables:	0	0	0	0	0	0	0	0	0
Penalty weights:				0	5	2	4	3	0

number of penalty points incurred by missing the goals

Min z= 0



Goal Programming - Nonpreemptive

Excel Solver:

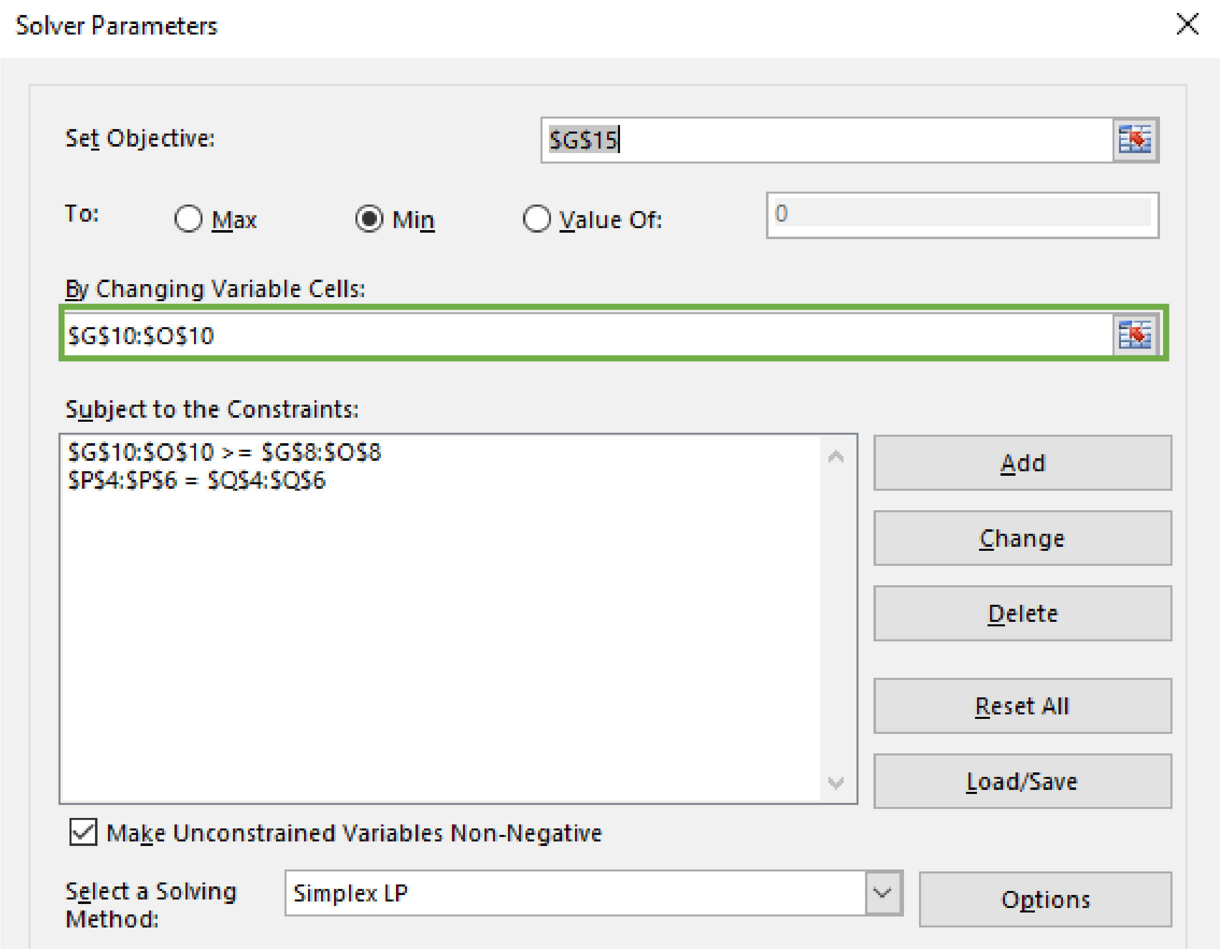
x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	0	125
employment	5	3	4	0	0	-1	1	0	0	0	40
investment	5	7	8	0	0	0	0	-1	1	0	55

	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-
lower bounds:	0	0	0	0	0	0	0	0	0
decision variables:	0	0	0	0	0	0	0	0	0
Penalty weights:				0	5	2	4	3	0

number of penalty points incurred by missing the goals

Min z= 0

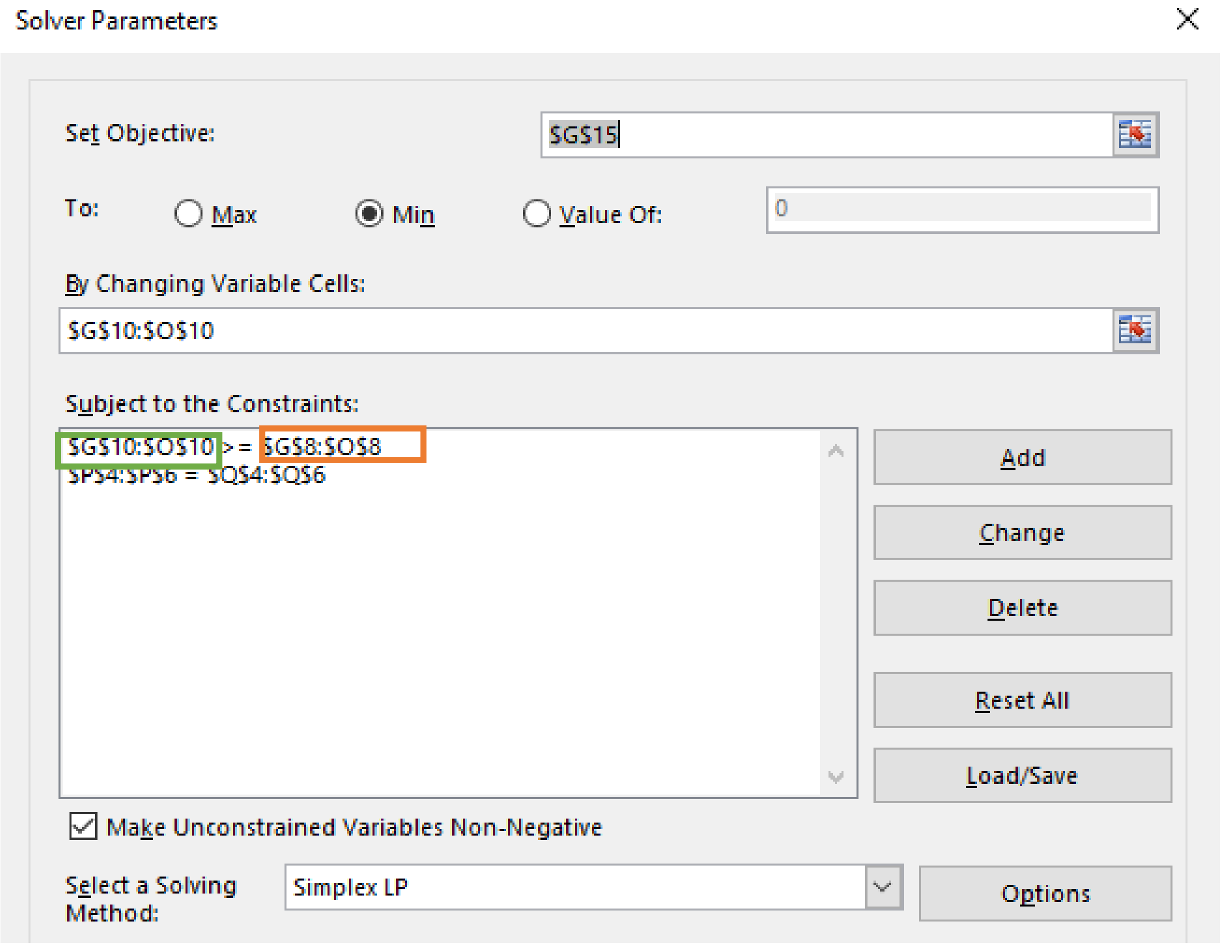


Goal Programming - Nonpreemptive

Excel Solver:

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	0	125
employment	5	3	4	0	0	-1	1	0	0	0	40
investment	5	7	8	0	0	0	0	-1	1	0	55
lower bounds:	0	0	0	0	0	0	0	0	0		
decision variables:	0	0	0	0	0	0	0	0	0		
Penalty weights:				0	5	2	4	3	0		
				number of penalty points incurred by missing the goals							
Min z=	0										



Goal Programming - Nonpreemptive

Excel Solver:

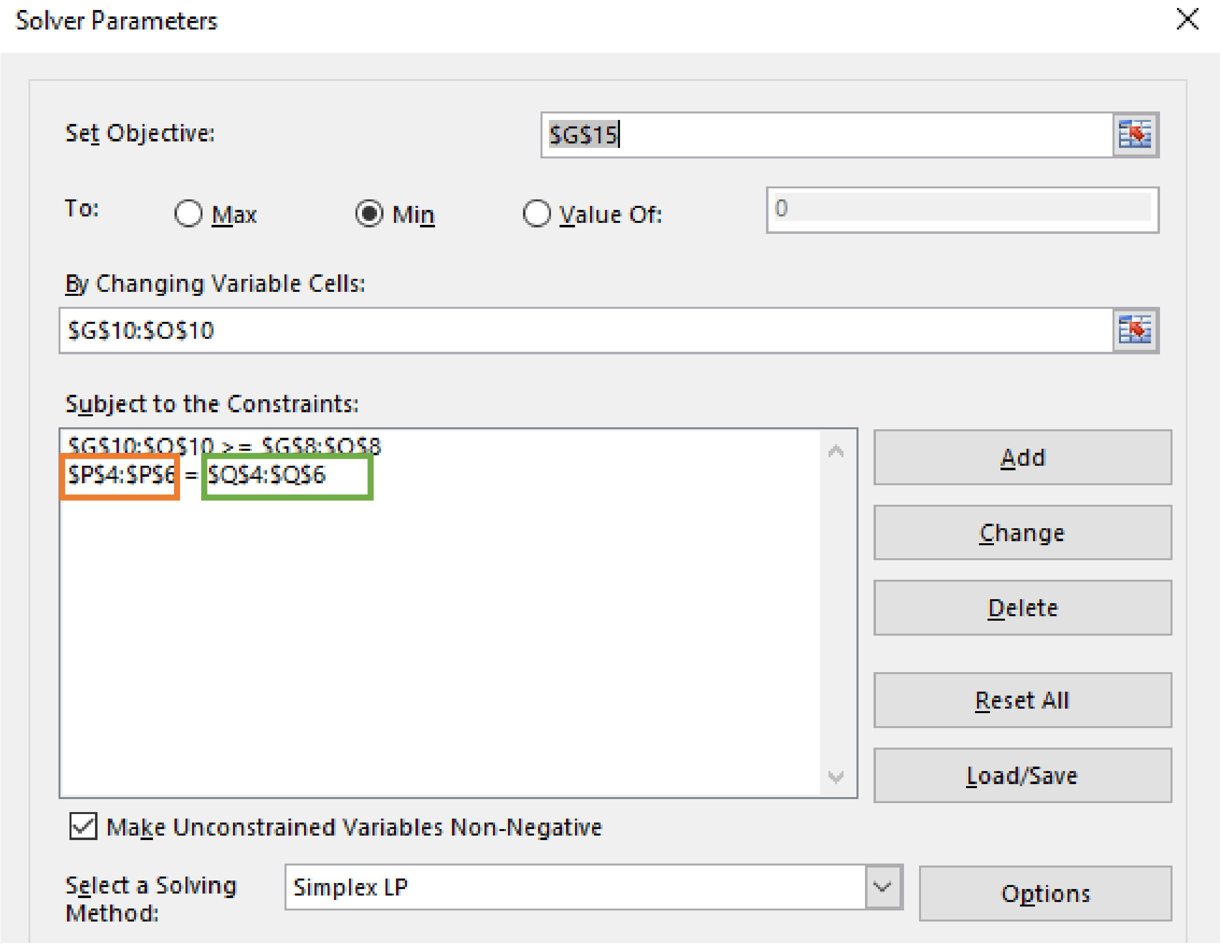
x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	0	125
employment	5	3	4	0	0	-1	1	0	0	0	40
investment	5	7	8	0	0	0	0	-1	1	0	55

	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-
lower bounds:	0	0	0	0	0	0	0	0	0
decision variables:	0	0	0	0	0	0	0	0	0
Penalty weights:				0	5	2	4	3	0

number of penalty points incurred by missing the goals

Min z= 0



Goal Programming - Nonpreemptive

Excel Solver:

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

	Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$	profit	12	9	15	-1	1	0	0	0	0	125	125
$5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) = 40$	employment	5	3	4	0	0	-1	1	0	0	40	40
$5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) = 55$	investment	5	7	8	0	0	0	0	-1	1	55	55
$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$	lower bounds:	0	0	0	0	0	0	0	0	0		
	decision variables:	8.33	0	1.67	0	0	8.33	0	0	0		
$\text{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$	Penalty weights:				0	5	2	4	3	0		

number of penalty points incurred by missing the goals

Min z= 16.7

Goal Programming - Nonpreemptive

Excel Solver:

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

	Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$	profit	12	9	15	-1	1	0	0	0	0	125	125
$5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) = 40$	employment	5	3	4	0	0	-1	1	0	0	40	40
$5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) = 55$	investment	5	7	8	0	0	0	0	-1	1	55	55
$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$	lower bounds:	0	0	0	0	0	0	0	0	0		
	decision variables:	8.33	0	1.67	0	0	8.33	0	0	0		
Min $Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$	Penalty weights:				0	5	2	4	3	0		

number of penalty points incurred by missing the goals

Min z= 16.7

This means we're 8.33 above the employment goal of 40 hundred (833 employees)

Goal Programming - Nonpreemptive

Excel Solver:

x1, x2, x3 are the production rates of products 1, 2, and 3, respectively

$$12x_1 + 9x_2 + 15x_3 - (y_1^+ - y_1^-) = 125$$

$$5x_1 + 3x_2 + 4x_3 - (y_2^+ - y_2^-) = 40$$

$$5x_1 + 7x_2 + 8x_3 - (y_3^+ - y_3^-) = 55$$

$$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$$

$$\textbf{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$$

Goals	x1	x2	x3	y1+	y1-	y2+	y2-	y3+	y3-	total	RHS
profit	12	9	15	-1	1	0	0	0	0	125	125
employment	5	3	4	0	0	-1	1	0	0	40	40
investment	5	7	8	0	0	0	0	-1	1	55	55
lower bounds:	0	0	0	0	0	0	0	0	0		
decision variables:	8.33	0	1.67	0	0	8.33	0	0	0		
Penalty weights:				0	5	2	4	3	0		

number of penalty points incurred by missing the goals

Min z= 16.7

This means the best way to achieve the goals given these penalties is not producing product X_2

Goal Programming - Nonpreemptive

Exercise 1:

A project manager wants to find the quantities of 3 products. Producing 1 unit of:

- product 1 requires **40 employees**, **2 tons of raw material** and will bring the company a **profit of 5 hundred €**
- product 2 requires **30 employees**, **4 tons of raw material** and will bring the company a **profit of 8 hundred €**
- product 3 requires **20 employees**, **3 tons of raw material** and will bring the company a **profit of 4 hundred €**

The manager has 3 goals:

1) The maximum number of employees that can be allocated to producing these 3 products is 100 employees
2) There are 10 tons of raw material in the warehouse and he wants to consume no more no less than that
3) The total profit is expected to be at least 30 hundred €

The manager suspects he might not be able to meet these 3 goals simultaneously therefore he sets some penalty weights to each of the goals:

- Each extra employee is associated to a penalty of 5
- Each ton below the goal is associated to a penalty of 8 (-) whereas each ton above the goal of 10 is associated to a penalty of 12 (+)
- If profit is less than 30 hundred €, each hundred € is associated to a penalty of 15

Formulate the problem as a linear programming problem and use excel solver (LP simplex) to find the combination of the 3 products that minimizes the penalties.

Goal Programming - Nonpreemptive

Exercise 2:

Reconsider the original version of the Dewright Co. After further reflection about the solution obtained by the simplex method, management now is asking some what-if questions.

(a) Management wonders what would happen if the penalty weights in the rightmost column of Table 7.5 were to be changed to 7, 4, 1, and 3, respectively. Would you expect the optimal solution to change? Why?

(b) Management is wondering what would happen if the total profit goal were to be increased to wanting at least $140 million (without any change in the original penalty weights). Solve the revised model with this change.

(c) Solve the revised model if both changes are made.

Goal Programming - Nonpreemptive

Exercise 2:

Factor	Unit Contribution Product: 1	 2	 3	Goal (Units)	Penalty Weight
Long-run profit	12	9	15	$\geq$ 125 (millions of dollars)	5
Employment level	5	3	4	= 40 (hundreds of employees)	2(+), 4(−)
Capital investment	5	7	8	$\leq$ 55 (millions of dollars)	3

- penalty weights in the rightmost column of Table 7.5 were to be changed to 7, 4, 1, and 3, respectively.
- profit goal were to be increased to wanting at least $140 million

Goal Programming - Preemptive

- In *non-preemptive* GP we assume that all goals are of roughly comparable importance.
- HOWEVER, when one or more of the goals clearly are far more important than the others (***preemptive* GP**), the initial focus should be on achieving as closely as possible these ***first-priority* goals**.
- The other goals also might naturally divide further into *second-priority* goals, *third-priority goals*, and so on.

Goal Programming - Preemptive

When a hierarchy of priority levels for the goals is considered:

find an optimal solution with for the *first-priority goals* —Re-optimize→ break any ties for the optimal solution by considering the *second-priority goals* —Re-optimize→ Remaining ties can be broken by considering the *third-priority goals*

- When we deal with goals on the *same* priority level, our approach is just like the one described for non-preemptive goal programming.
- Any of the same three types of goals (lower one-sided, two-sided, upper one-sided) can arise.
- Different penalty weights for deviations from different goals still can be included, if desired.
- The same formulation technique of introducing auxiliary variables again is used to reformulate this portion of the problem to fit the linear programming format.

Goal Programming - Preemptive

- There are two basic methods based on linear programming for solving preemptive goal programming problems:
  - *sequential procedure*
  - *streamlined procedure*

Let us illustrate these procedures by solving an exemple:

Goal Programming – Preemptive

The Dewright Company has reconsidered the original formulation of the problem (summarized in the table) to face the recommendation to increase the company's workforce by more than 20 percent

	Unit Contribution				
	Product:				
Factor	**1**	**2**	**3**	**Goal (Units)**	**Penalty Weight**
Long-run profit	12	9	15	≥ 125 (millions of dollars)	5
Employment level	5	3	4	= 40 (hundreds of employees)	2(+), 4(−)
Capital investment	5	7	8	≤ 55 (millions of dollars)	3

This probably would be a temporary increase, so the very high cost of training 833 new employees would be largely wasted

Management has concluded that a very high priority should be placed on:

- avoiding an increase in the workforce.
- avoiding capital investment above $55 million

Goal Programming - Preemptive

Based on these considerations, a ***preemptive goal programming*** approach should now be used, where the first-priority goals should be:

- a very high priority should be placed on avoiding capital investment
- avoiding an increase in the workforce

and the other two original goals (the second priority goals):

- exceeding $125 million in long-run profit
- avoiding a decrease in the employment level

Within the two priority levels, management feels that the relative penalty weights still should be the same as those given in the rightmost column of the table.

Goal Programming - Preemptive

This reformulation is summarized below:

Priority Level	Factor	Goal	Penalty Weight
First priority	Employment level	≤ 40	$2M$
	Capital investment	≤ 55	$3M$
Second priority	Long-run profit	≥ 125	5
	Employment level	≥ 40	4

where a factor of M (representing a huge positive number) has been included in the penalty weights for the *first-priority* goals to emphasize that these goals preempt the *second-priority* goals.

The portions of the original table not included in this one remained unchanged

Goal Programming - Preemptive

The Sequential Procedure

- The preemptive GP problem consists of **solving a *sequence* of linear programming models**.

- 1st stage –only the first-priority goals goals included in the linear programming model are considered
- 2nd stage - the simplex method is applied in the usual way

If the resulting **optimal solution is *unique***,

we adopt it immediately disregarding any additional goals.

If there are ***multiple* optimal solutions** with the same optimal value of *Z* (call it *Z**),

we prepare to break the tie among these solutions by moving to the second stage and adding the second-priority goals to the model.

Goal Programming - Preemptive

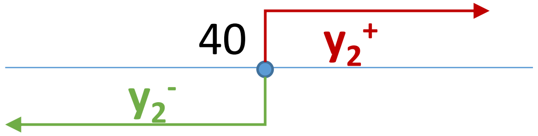
At the first stage, only the two ***first-priority* goals** are included in the linear programming model. Therefore, we can drop the common factor M for their penalty weights.

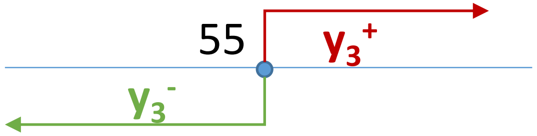
By proceeding just as for the non-preemptive model if these were the only goals, the resulting linear programming model is

$$\text{Minimize } Z = 2\, y2^{+} + 3\, y3^{+}$$

subject to:

$$5x1 + 3x2 + 4x3 - (y2^{+} - y2^{-}) \leq 40$$
$$5x1 + 7x2 + 8x3 - (y3^{+} - y3^{-}) \leq 55$$





And $xj \geq 0$, $yk^{+} \geq 0$, $yk^{-} \geq 0$ (j = 1, 2, 3; k = 2, 3)

For ease of comparison with the non-preemptive model with all four goals, we have kept the same subscripts on the auxiliary variables.

Goal Programming - Preemptive

By using the simplex method, an optimal solution for this linear programming model has $y2^+ = 0$ and $y3^+ = 0$, with $Z = 0$ (so $Z^* = 0$), because there are innumerable solutions for ($x1$, $x2$, $x3$) that satisfy the relationships

$$5x1 + 3x2 + 4x3 \leq 40$$

$$5x1 + 7x2 + 8x3 \leq 55$$

as well as the non-negativity constraints.

Therefore, these two *first-priority goals* should be used as *constraints* hereafter.

Using them as constraints will force $y2^+$ and $y3^+$ to remain zero and thereby disappear from the model automatically.

Goal Programming - Preemptive

By using the simplex method, an optimal solution for this linear programming model has $y2^+ = 0$ and $y3^+ = 0$, with $Z = 0$ (so $Z^* = 0$), because there are innumerable solutions for ($x1$, $x2$, $x3$) that satisfy the relationships

$$5x1 + 3x2 + 4x3 \leq 40$$
$$5x1 + 7x2 + 8x3 \leq 55$$

as well as the non-negativity constraints.

Therefore, these two *first-priority goals* should be used as *constraints* hereafter.

Using them as constraints will force $y2^+$ and $y3^+$ to remain zero and thereby disappear from the model automatically.

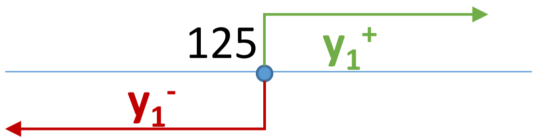
Goal Programming - Preemptive

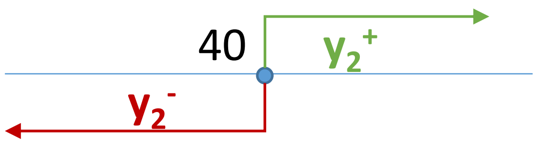
If we drop $y2^+$ and $y3^+$ but add the *second-priority goals*, the second-stage linear programming model becomes

$$\text{Minimize } Z = 5y1^- + 4y2^-$$

subject to:

$$12x1 + 9x2 + 15x3 - (y1^+ - y1^-) = 125$$
$$5x1 + 3x2 + 4x3 + y2^- = 40$$
$$5x1 + 7x2 + 8x3 + y3^- = 55$$





And $xj \geq 0,\ y1 \geq 0,\ yk \geq 0$ ($j = 1, 2, 3$; $k = 1, 2, 3$)

Applying the simplex method to this model yields the unique optimal solution $x1 = 5$, $x2 = 0$, $x3 = 3\ 3/4$, $y1^+ = 0$, $y1^+ = 8\ 3/4$, $y2^- = 0$, and $y3^- = 0$, with $Z = 43\ ¾$.

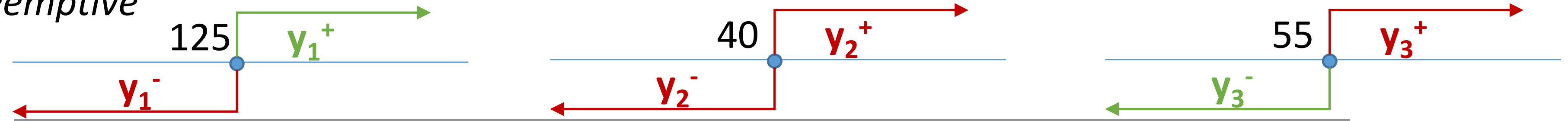
Goal Programming - Preemptive

Because <u>this solution is unique </u>(*or* because there are no more priority levels), <u>the procedure can now stop</u>, with (*x*1, *x*2, *x*3) (5, 0, 3 3/4) as the optimal solution for the *overall* problem.

This solution fully achieves both *first-priority goals* as well as one of the second-priority goals (no decrease in employment level), and it falls short of the other *second-priority goal* (long-run profit 125) by just 8 3/4.

Goal Programming – Preemptive

Non-preemptive

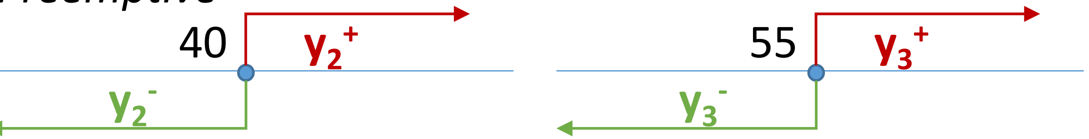


Factor	Unit Contribution Product: 1	2	3	Goal (Units)	Penalty Weight
Long-run profit	12	9	15	≥ 125 (millions of dollars)	5
Employment level	5	3	4	= 40 (hundreds of employees)	2(+), 4(−)
Capital investment	5	7	8	≤ 55 (millions of dollars)	3

Min $Z = 5y_1^- + 2\,y_2^+ + 4\,y_2^- + 3\,y_3^+$

$$\begin{aligned} 12x_1 + 9x_2 + 15x_3 &\geq 125 \\ 5x_1 + 3x_2 + 4x_3 &= 40 \\ 5x_1 + 7x_2 + 8x_3 &\leq 55 \end{aligned}$$

Preemptive



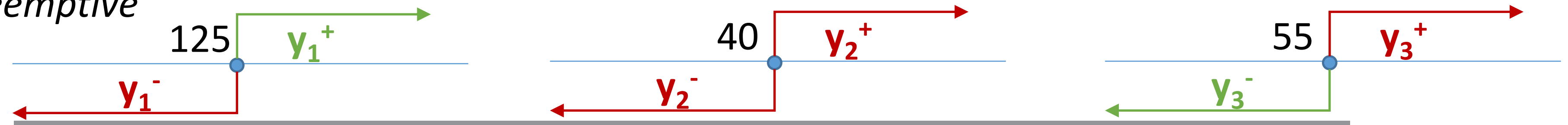
Priority Level	Factor	Goal	Penalty Weight
First priority	Employment level	≤40	2*M*
	Capital investment	≤55	3*M*
Second priority	Long-run profit	≥125	5
	Employment level	≥40	4

Minimize $Z = 2y2^+ + 3y3^+$

$$\begin{aligned} 5x_1 + 3x_2 + 4x_3 &\leq 40 \\ 5x_1 + 7x_2 + 8x_3 &\leq 55 \end{aligned}$$

Goal Programming – Preemptive

Non-preemptive



Factor	Unit Contribution Product: 1	2	3	Goal (Units)	Penalty Weight
Long-run profit	12	9	15	$\geq$ 125 (millions of dollars)	5
Employment level	5	3	4	= 40 (hundreds of employees)	2(+), 4(−)
Capital investment	5	7	8	$\leq$ 55 (millions of dollars)	3

$$\text{Min } Z = 5y_1^- + 2y_2^+ + 4y_2^- + 3y_3^+$$

$$12x_1 + 9x_2 + 15x_3 \geq 125$$
$$5x_1 + 3x_2 + 4x_3 = 40$$
$$5x_1 + 7x_2 + 8x_3 \leq 55$$

Preemptive

Priority Level	Factor	Goal	Penalty Weight
First priority	Employment level	$\leq$40	2*M*
	Capital investment	$\leq$55	3*M*
Second priority	Long-run profit	$\geq$125	5
	Employment level	$\geq$40	4

Minimize $Z = 2y2^+ + 3y3^+$

$$5x_1 + 3x_2 + 4x_3 \leq 40$$
$$5x_1 + 7x_2 + 8x_3 \leq 55$$

Minimize $Z = 5y1^- + 4y2^-$

$$12x_1 + 9x_2 + 15x_3 \geq 125$$
$$5x_1 + 3x_2 + 4x_3 \geq 40$$

Goal Programming - Preemptive

the *streamlined procedure*, instead of solving a sequence of linear programming models, finds an optimal solution by solving just *one* linear programming model.

Thus, the streamlined procedure is able to duplicate the work of the sequential procedure with just *one run* of the simplex method.

This one run *simultaneously* finds optimal solutions based just on *first-priority goals* and breaks ties among these solutions by considering lower-priority goals.

However, this does require a slight modification of the simplex method.

Goal Programming - Preemptive

If there are just *two* priority levels, the modification of the simplex method consists of applying the **b*ig M method***

In this form, instead of replacing *M* throughout the model by some huge positive number before running the simplex method, we retain the *symbolic* quantity *M* in the sequence of simplex tableaux.

Each coefficient in row 0 (for each iteration) is some linear function ***aM+b***, where *a* is the current *multiplicative factor* and *b* is the current *additive term.*

Goal Programming - Preemptive

The **usual decisions** based on these coefficients (entering basic variable and optimality now **are based solely on the *multiplicative* factors**, except that any ties would be broken by using the *additive* terms.

This is how the OR Courseware operates when solving interactively by the simplex method (and choosing the **Big *M* method**).

The LP formulation with **two priority levels** would:

- include *all* the goals in the model (in the usual manner),
- basic penalty weights of M assigned to deviations from first-priority goals
- basic penalty weights of 1 assigned to deviations from *second-priority* goals

If different penalty weights are desired within the same priority level, these basic penalty weights then are multiplied by the individual penalty weights assigned within the level.

Goal Programming - Preemptive

Example:

For the goal programming problem summarized below, note that

- (1) different penalty weights are assigned within each of the two priority levels
- (2) the individual penalty weights (2 and 3) for the first-priority goals have been multiplied by M

Priority Level	Factor	Goal	Penalty Weight
First priority	Employment level	≤40	$2M$
	Capital investment	≤55	$3M$
Second priority	Long-run profit	≥125	5
	Employment level	≥40	4

Goal Programming - Preemptive

Example:

For the goal programming problem summarized below, note that

Priority Level	Factor	Goal	Penalty Weight
First priority	Employment level	≤ 40	$2M$
	Capital investment	≤ 55	$3M$
Second priority	Long-run profit	≥ 125	5
	Employment level	≥ 40	4

Minimize $Z = 5y1^{-} + 2My2^{+} + 4y2^{-} + 3My3^{+}$,

subject to

$$12x1 + 9x2 + 15x3 - (y1^{+} - y1^{-}) = 125$$
$$5x1 + 3x2 + 4x3 - (y2^{+} - y2^{-}) = 40$$
$$5x1 + 7x2 + 8x3 - (y3^{+} - y3^{-}) = 55$$

And $xj \geq 0$, $yk \geq 0$, $yk \geq 0$ (j 1, 2, 3; k 1, 2, 3).

Goal Programming - Preemptive

Because this model uses *M* to symbolize a huge positive number, the simplex method can be applied as described in previous classes.

Alternatively, a in the model and then any software pac*very large positive number can be substituted for M* kage based on the simplex method can be applied.

Doing either naturally yields **the same unique optimal** solution obtained by the sequential procedure.

Goal Programming - Preemptive

The LP formulation with **more than two priority levels** can be generalized in a straightforward way:

The basic penalty weights for the respective levels now are M1, M2, . . . , Mp-1, 1.

Where:

M1 represents a number that is vastly larger than M2,
M2 is vastly larger than M3, . . . , and
Mp-1 is vastly larger than 1.

Each coefficient in row 0 of each simplex tableau is now a linear function of all of these quantities, where the multiplicative factor of M1 is used to make the necessary decisions, with tie breakers beginning with the multiplicative factor of M2 and ending with the additive term.

Goal Programming - Preemptive

The LP formulation with **more than two priority levels** can be generalized in a straightforward way:

The basic penalty weights for the respective levels now are M1, M2, . . . , Mp-1, 1.

Where:

M1 represents a number that is vastly larger than M2,
M2 is vastly larger than M3, . . . , and
Mp-1 is vastly larger than 1.

Each coefficient in row 0 of each simplex tableau is now a linear function of all of these quantities, where the multiplicative factor of M1 is used to make the necessary decisions, with tie breakers beginning with the multiplicative factor of M2 and ending with the additive term.

Goal Programming

Conclusions:

Linear goal programming and its solution procedures provide an effective way of dealing with problems where management wishes to **strive toward several goals simultaneously**

The key is a formulation technique of introducing auxiliary variables that enable converting the problem into a linear programming format.

Goal Programming - Preemptive

- If $Z^*=0$

all the auxiliary variables representing the *deviations from first-priority goals* must equal zero (full achievement of these goals)

Thus, all these auxiliary variables now can be completely deleted from the model, where the equality constraints that contain these variables are replaced by the mathematical expressions (inequalities or equations) for these first-priority goals, to ensure that they continue to be fully achieved.

- if $Z^* > 0$

the second-stage model simply adds the second-priority goals to the first-stage model (as if these additional goals actually were first-priority goals),

but then it also adds the constraint that the *first-stage objective function* equals Z^* (which enables us again to delete the terms involving first-priority goals from the second-stage objective function).

After we apply the simplex method again, if there still are multiple optimal solutions, we repeat the same process for any lower priority goals.