# Ex 2.42

2 de novembro de 2020 10:17

**2.42.** Seja $(X,Y)$ o par aleatório com a seguinte função densidade de probabilidade conjunta:

$$f(x,y) = \begin{cases} a\,(x+y) & 1 \le x \le 2 \quad 1 \le y \le 2 \\ 0 & \text{restantes valores de } (x,y). \end{cases}$$

a) Determine o valor da constante $a$.

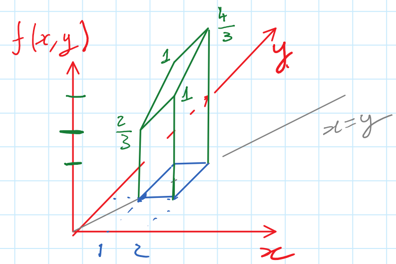
b) $X$ e $Y$ são variáveis aleatórias independentes? Justifique.

c) Calcule $P[Y < X]$ e $P[Y > 3 - X]$. Comente.

d) Determine o valor médio de $1/X$.

e) Determine a função densidade condicional de $Y \mid X = 3/2$.

f) Calcule $P[Y < 3/2 \mid X = 3/2]$.




$$f(x,y) = \begin{cases} a(x+y) & , \ 1 \le x \le 2 \ , \ 1 \le y \le 2 \\ 0 & , \ \text{outros valores} \end{cases}$$

a) COMO $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$, ENTÃO

$$\int_1^2 \int_1^2 (x+y)\,dx\,dy = a \int_1^2 \left[\frac{x^2}{2} + xy\right]_{x=1}^{x=2} dy$$

$$= \int_1^2 \left(\frac{4}{2} + 2y - \frac{1}{2} - y\right) dy = a \int_1^2 \left(\frac{3}{2} + y\right) dy =$$

$$= a \left[\frac{3}{2} y + \frac{y^2}{2}\right]_1^2 = \left(\frac{3}{2} \times 2 + \frac{4}{2} - \frac{3}{2} - \frac{1}{2}\right) =$$

$$= a \times 3 = 1 \quad \text{OU SEJA} \quad a = \frac{1}{3}$$

b) X, Y VA INDEPENDENTES SSE $\underline{f(x,y)} = f_X(x) \times f_Y(y)$

... de Y

função densidade marginal de X

$$f_X(x) = \begin{cases} \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_1^2 = \\ = \frac{1}{3}\left(2x + 2 - x - \frac{1}{2}\right) = \frac{1}{3}\left(x + \frac{3}{2}\right) = \frac{1}{3}x + \frac{1}{2} \ \text{SE} \ 1 \le x \le 2 \end{cases}$$

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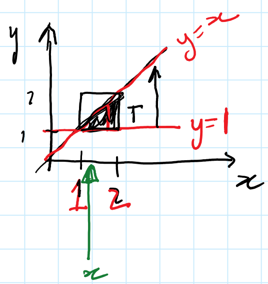
$$f_Y(y) = \begin{cases} \frac{1}{3}y + \frac{1}{2}, & 1 \leq y \leq 2 \\ 0, & \text{OUTROS VALORES} \end{cases}$$

EXACTAMENTE OS MESMOS CÁLCULOS TROCANDO $x$ E $y$

$$f_X(x) \cdot f_Y(y) = \begin{cases} \left(\frac{1}{3}x + \frac{1}{2}\right)\left(\frac{1}{3}y + \frac{1}{2}\right), & 1<x<2,\ 1<y<2 \\ 0, & \text{OUTROS VALORES} \end{cases}$$

COMO $f(x,y) \neq f_X(x)\, f_Y(y)$ ENTÃO $X, Y$ NÃO SÃO INDEPENDENTES.

c) $P(Y<X)$




$$P(Y<X) = \iint_T f(x,y)\,dx\,dy = \iint_T \frac{1}{3}(x+y)\,dx\,dy$$

$$= \int_1^2 \left( \int_{y=1}^{y=x} \frac{1}{3}(x+y)\,dy \right) dx \underset{C.A}{=} \int_1^2 \left\{ \frac{x^2}{2} - \frac{x}{3} - \frac{1}{6} \right\} dx$$

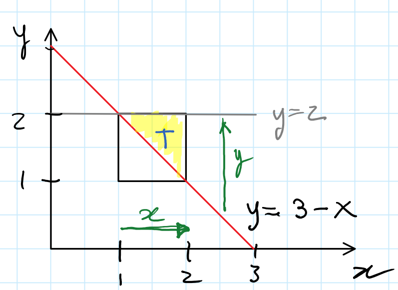
$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 - \frac{1}{3}\left[\frac{x^2}{2}\right]_1^2 - \frac{1}{6}(2-1) = \frac{1}{2}\left(\frac{8}{3}-\frac{1}{3}\right) - \frac{1}{3}\left(\frac{4}{2}-\frac{1}{2}\right) - \frac{1}{6}$$

$$= \frac{1}{2}\frac{7}{3} - \frac{1}{2} - \frac{1}{6} = \frac{1}{2}$$

COMO SERIA DE ESPERAR DADA A SIMETRIA DE $f(x,y)$ RELATIVAMENTE À RECTA $y=x$.

C.A. $$\int_{y=1}^{y=x} \frac{1}{3}(x+y)\,dy = \frac{1}{3}\left[xy + \frac{y^2}{2}\right]_{y=1}^{y=x} = \frac{1}{3}\left(\left(x^2 + \frac{x^2}{2}\right) - \left(x + \frac{1}{2}\right)\right) = \frac{x^2}{2} - \frac{x}{3} - \frac{1}{6}$$

$P(Y > 3-X)$




$$P(Y>3-X) = \iint_T f(x,y)\,dx\,dy = \int^2 \left( \int^2 \frac{1}{3}(x+y)\,dy \right) dx = *$$

$$P(Y>3-X) = \iint_T f(x,y)\,dx\,dy = \int_1^2 \left( \int_{y=3-x}^{2} \frac{1}{3}(x+y)\,dy \right) dx \underset{C.A.}{=} *$$

$$C.A.\ \frac{1}{3}\int_{y=3-x}^{2} (x+y)\,dy = \frac{1}{3}\left[ xy + \frac{y^2}{2} \right]_{y=3-x}^{y=2} = \frac{1}{3}\left( 2x + \frac{4}{2} - x(3-x) - \frac{(3-x)^2}{2} \right) =$$

$$= \frac{1}{3}\left(2x + 2 - 3x + x^2 - \frac{9}{2} - \frac{x^2}{2} + 3x\right) = \frac{1}{3}\left(\frac{x^2}{2} + 2x - \frac{5}{2}\right)$$

$$* = \frac{1}{3}\int_1^2 \left(\frac{x^2}{2} + 2x - \frac{5}{2}\right) dx = \frac{1}{3}\left[\frac{x^3}{6} + x^2 - \frac{5}{2}x\right]_1^2 = \frac{1}{3}\left(\frac{8}{6} + 4 - 5 - \frac{1}{6} - 1 + \frac{5}{2}\right)$$

$$= \frac{1}{3}\left(\frac{7}{6} + 3 - \frac{5}{2}\right) = \frac{1}{3}\left(\frac{7+18-15}{6}\right) = \frac{1}{3} \times \frac{10}{6} = \frac{10}{18} = \frac{5}{9}.$$

NOTA: SERIA DE ESPERAR QUE $P(Y>3-X)$ FOSSE UM POUCO SUPERIOR A 0.5.

d) $$E\left[\frac{1}{X}\right] = \int_{-\infty}^{+\infty} \frac{1}{x} f_X(x)\,dx$$

Vimos que $f_X(x) = \begin{cases} \frac{1}{3}x + \frac{1}{2}, & \text{SE } 1 \leq x \leq 2 \\ 0, & \text{OUTROS VALORES DE } x \end{cases}$

$$E[X] = \int_1^2 \frac{1}{x}\left(\frac{1}{3}x + \frac{1}{2}\right) dx = \int_1^2 \left(\frac{1}{3} + \frac{1}{2x}\right) dx =$$

$$= \left[\frac{1}{3}x + \frac{1}{2}\ln x\right]_1^2 = \frac{2}{3} + \frac{1}{2}\ln 2 - \frac{1}{3} - \frac{1}{2}\underbrace{\ln 1}_{=0} = \frac{1}{3} + \frac{1}{2}\ln 2 = 0.68$$

e) $$f_{Y/X=3/2}(y) = \begin{cases} \left.\dfrac{f(x,y)}{f_X(x)}\right|_{x=3/2} = \left.\dfrac{\frac{1}{3}(x+y)}{\left(\frac{1}{3}x + \frac{1}{2}\right)}\right|_{x=3/2} = \dfrac{y}{3} + \dfrac{1}{2}, & 1 \leq y \leq 2 \\ 0, & \text{PARA OUTROS VALORES DE } y \end{cases}$$

f) $$P\left(Y < \frac{3}{2} \,/\, X = \frac{3}{2}\right) = \int_1^{3/2} f_{Y/X=3/2}(y)\,dy = \int_1 \left(\frac{y}{3} + \frac{1}{2}\right) dy = \left[\frac{y^2}{6} + \frac{1}{2}y\right]_1^{3/2} =$$

$$= \frac{\frac{9}{4}}{6} + \frac{1}{2}\frac{3}{2} - \frac{1}{6} - \frac{1}{2} = \frac{9}{24} + \frac{3}{4} - \frac{1}{6} - \frac{1}{2}$$

$$= \frac{9 + 18 - 4 - 12}{24} = \frac{11}{24}$$