

Allometric equations and growth functions

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1	Introduction.....	2
2	Empirical versus mechanistic or theoretical growth functions.....	3
3	Growth functions of the Lundqvist-Korf type	4
3.1	Schumacher's function	4
3.2	Lundqvist-Korf's function.....	5
4	Growth functions of the Richards type	6
4.1	Monomolecular function	6
4.2	Logistic and generalized logistic functions	7
4.3	Gompertz function.....	8
4.4	Richards function	8
5	Functions of the Hossfeld IV type.....	10
5.1	The Hossfeld IV function	10
5.2	McDill-Amateis/Hossfeld IV function.....	11
5.3	Generalizations of the Hossfeld IV function	12
6	Other growth functions	13
7	Zeide decomposition of growth functions	13
8	Formulating growth functions as a difference equation without t explicit	14
9	References.....	16

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NOTE: Text on allometric equations still to be added

1 Introduction

Growth functions describe the change in size of an individual or population with time. The selection of appropriate growth functions for tree and stand modeling is an important stage in the development of growth and yield models. Here we present information on the forms and characteristics of the more commonly-used growth functions for modeling forest development. When fitted to data, a number of these functions will give essentially equivalent results within the range of the observations used for estimating the equation's coefficients. However, their behavior when extrapolated may be quite different depending on the underlying mathematical properties involved. Hence, understanding these properties is helpful to modelers to determine which candidate functions to consider for specific applications.

Unless the data available for modeling cover a very small range of time, there are some properties that a growth function must fulfill so that it has a shape compatible with the principles of biological growth (Fig. 6.1):

- i) The curve must be limited by the value zero at a specific beginning ($t=0$ or $t=t_0$), depending if the variable that is being modeled starts in $t=0$, as is the case for the great majority of the tree and stand variables, or later on, as happens with tree diameter at breast height or stand basal area;
- ii) The curve should exhibit a maximum value usually achieved at an older age (existence of an asymptote);
- iii) The relative growth rate (variation of the variable x by unit of time and unit of x) should show a maximum in an initial phase and decreases thereafter; for most tree and stand forest variables, the maximum occurs very early and therefore the relative growth rate may be modeled using a decreasing function;
- iv) The slope of the curve should increase with increasing growth rate in the initial phase and decrease in the final stages (show an inflection point).

At this point it is important to understand the concepts of growth and yield. Growth is the increase in size of an individual or population per unit of time (for instance volume growth in $\text{m}^3\text{ha}^{-1}\text{yr}^{-1}$) while yield is the size of the tree or population at a certain point in time (for instance total volume at the age 50 in m^3ha^{-1}). In terms of continuous growth functions, yield corresponds to the integral form of the growth function and growth to its differential form. A yield function can be obtained by integrating the growth equation and a growth equation can be obtained by differentiating the yield equation.

2 Empirical versus mechanistic or theoretical growth functions

Functions used to model growth have been classified into one of two broad divisions (Thornley, 1976; Hunt, 1982; Vanclay, 1994), empirical and theoretical or mechanistic models. The last are conceived in terms of the mechanism of the system (Thornley, 1976), usually having an underlying hypothesis associated with the cause or function of the phenomenon described by the response variable (Vanclay, 1994), while the empirical models describe the behavior of the response variable without trying to identify the causes and explaining the phenomenon. The distinction between the two is not sharp and most modeling exercises contain both empiricism and mechanism in varying mixtures. According to Thornley (1976), the mechanistic modeler will tend to construct models before doing the experiments or analyzing the data, thinking of possible mechanisms and deducting their consequences by means of a model while the empirical modeler will describe the behavior of the response variable based on the data. If a mathematical model has a theoretical/biological basis, the parameter estimates, even if obtained from empirical data, can provide insight into the phenomenon that is being modeled.

Many equations have been used to model tree and stand growth, usually under the integral form. Table 6.1 summarizes some non-sigmoid functions that were often used in forest modeling (Grosenbaugh 1965; Prodan 1968). A longer list can be seen in Kiviste et al. (1988, 2002)ⁱ. None of these functions exhibit all the desirable properties listed previously; thus, when using them for modeling, one needs to be cautious with extrapolations outside the

ⁱ The analysis of growth functions published by Kiviste (1988) in Russian was later partially translated into Spanish (Kiviste et al. 2002). We are not aware of an English version of this work.

range of data used to fit the models and with the signs that the parameters can take so that they exhibit a shape compatible with biological growth.

The functions known as Freese, Hossfeld I and Korsun show extremes (maximum and/or minimum) that, depending on the signs of the coefficients, may follow within the range of time values relevant for forest growth modeling. However they follow approximately the shape of a growth curve for a limited range of ages and have been used in several growth modeling applications, especially the Hossfeld I function.

Theoretical growth functions have usually been developed in their growth form – either absolute or relative growth – and the respective yield form has been obtained by integration. In most of the cases this method allows interpretation of the function parameters and helps to impose restrictions on the values that the parameters can take to be biologically consistent.

Several authors (e.g. Grosenbaugh, 1965; Pienaar and Turnbull, 1973; Causton and Venus, 1981; Hunt, 1982; Zeide, 1993; Kiviste et al., 2002) have analysed theoretical growth functions properties in connection to the meaning of their parameters. Table 6.2 summarizes the properties of the sigmoid growth functions that are more commonly used in forest modeling. In this table, the asymptote is designated by A , the k parameter is related to the slope of the curve (growth rate), the m parameter is a shape parameter and usually the c parameter relates to the initial condition used. An analysis of the assumptions from which the functions in Table 6.2 were derived, as well as the meaning of the parameters, is given in the following sections. In these sections the functions are grouped according to their functional form, the groups being designated by the most general or known function of the group: 1) Lundqvist-Korf; 2) Richards; 3) Hossfeld IV/McDill-Amateis; 4) other growth functions.

3 Growth functions of the Lundqvist-Korf type

3.1 Schumacher's function

Schumacher's function (Schumacher, 1939) represents an early attempt in forestry to develop a growth function from biologically sound assumptions. The model proposed by this author for "generalized use", relies on the hypothesis that the relative growth rate increases linearly with the squared inverse of time (which means that it decreases nonlinearly with time):

$$\frac{1}{Y} \frac{dY}{dt} = k \frac{1}{t^2} \quad (6.1)$$

The yield function comes as

$$Y = A e^{-k \frac{1}{t}} \quad (6.2)$$

where $A = Y_0 e^{k/t_0}$ is the asymptote and (t_0, Y_0) are the initial values.

The k parameter expresses the rate of decrease of the relative growth rate and is therefore inversely related to the growth rate. The location of the inflection point depends on the value of k ; the value of Y at the time at which the inflection occurs depends on A and k .

The Johnson-Schumacher's function (Grosenbaugh, 1965) is a generalization of Schumacher's function that includes the cases when the initial yield for $t=0$ is not zero.

$$Y = A e^{-k \frac{1}{t+b}}$$

The additional parameter implies that, for $t=0$, the tree/stand has already attained the dimension $Ae^{-k/b}$. The addition of an extra parameter is of limited value except in the cases in which the trees are planted and one wants to consider the initial size of the seedlings.

3.2 Lundqvist-Korf's function

Another generalization is the Lundqvist-Korf's function (Korf, 1939; Lundqvist, 1957):

$$\frac{1}{Y} \frac{dY}{dt} = k \frac{m}{t^{(m+1)}} \quad (6.3)$$

The corresponding yield function is:

$$Y = A e^{-k \frac{1}{t^m}} \quad (6.4)$$

where $A = Y_0 e^{k/t_0^m}$ is the asymptote and (t_0, Y_0) are the initial values.

The k parameter is inversely related to the growth rate but this is also influenced by the m parameter, adding flexibility to the curve.

Figure 6.2 illustrates the flexibility of the Lundqvist function. By changing the parameters (asymptote, k and m) it is possible to cover a large range of shapes. When fixing the other parameters, the k parameter has an inverse relationship with the growth rate, while the reverse is true for the m parameter. It is important to stress that the 3 parameters interact as is shown on Fig. 6.2-C; it is possible to obtain higher growth with a smaller value of m , if the latter is combined with a higher asymptote. The location of the inflexion point does not depend on the value of the asymptote, but on the combined values of k and m , increasing with k . The effect of m is highly dependent on the k value. Figure 6.3 shows the combined effect of the 2 shape parameters on the age at which the inflexion occurs. The respective Y value, however, does not depend on the k parameter, but on the asymptote and m parameter values (Fig. 6.4), occurring at higher Y values, the higher the asymptote and the m value.

4 Growth functions of the Richards type

4.1 Monomolecular function

The monomolecular function, sometimes referred in agriculture and economics as the Mitscherlich function or law of diminishing returns (Zeide, 1993), can be obtained under the assumption that the absolute growth rate is proportional to the difference between the maximum value (asymptote) and the present dimension:

$$\frac{dY}{dt} = k(A - Y) = kA - kY \quad (6.5)$$

This relationship means that the absolute growth rate decreases linearly with the size of the individual or population (Y).

The yield form is

$$Y = A \left(1 - c e^{-k t} \right), \quad (6.6)$$

with $c = e^{k t_0} \left(1 - \frac{Y_0}{A} \right)$

Using the initial condition $Y(0)=0$ leads to $c=1$.

The function has an upper asymptote A , but no inflection point. The k parameter expresses the rate of decrease of the absolute growth rate.

4.2 Logistic and generalized logistic functions

The logistic function, first developed for population's growth, is one of the most well-known sigmoid functions. It has been applied to the volume yield of loblolly pine in 1937 (MacKinney et al., 1937). It is based on the assumption that the relative growth rate is equal to a biotic potential k , reduced according to the size/dimension of the population (in the present case tree/stand):

$$\frac{1}{Y} \frac{dY}{dt} = (k - mY) \quad (6.7)$$

The relative growth rate is therefore a declining linear function of the dimension.

The yield function is

$$Y = \frac{A}{\left(1 + c e^{-k t}\right)} \quad (6.8)$$

with $c = \frac{Y_0/m}{k - mY_0} e^{kY_0}$ and $A = \frac{k}{m}$

The inflection point of the logistic function occurs at $t = \log(c)/k$ and $Y = A/2$ and the curve is symmetric around the inflection point. The parameter k is the maximum relative growth rate and corresponds to the initial stage of growth.

The generalization of the logistic function (Pearl and Reed, 1923) had the objective to overcome the symmetry of the logistic curve. The most usual integral form of the generalized function is:

$$Y = \frac{A}{\left(1 + c e^{-(a_1 t + a_2 t^2 + a_3 t^3)}\right)}$$

where a_1 , a_2 and a_3 are parameters that define the shape of the curve. The inflection point is variable and almost incalculable. Note that the function may have more than one inflection

point. A detailed study of this function as well as a reparametrization can be seen in Grosenbaugh (1965).

Monserud (1984) proposed the following generalization of the logistic growth function:

$$Y = \frac{A}{(1 + c e^{-f(\mathbf{X}, t)})} \quad (6.9)$$

where $f(\mathbf{X}, t)$ is a function of age (t) and of several independent variables (\mathbf{X}). A is the asymptote and c is the half-saturation parameter that defines the value of $e^{-f(\mathbf{X}, t)}$ at which $Y(t) = A/2$ (Cieszewski, 2002).

4.3 Gompertz function

The Gompertz equation (Gompertz, 1825) was designed to describe age distribution in human populations and later on applied as a growth model (Winsor, 1932). This function can be directly obtained from the following differential equation:

$$\frac{1}{Y} \frac{dY}{dt} = k (\log A - \log Y) = -k \log\left(\frac{Y}{A}\right) \quad (6.10)$$

The function assumes that the relative growth rate is inversely related to the logarithm of the ratio between the present dimension of Y and the respective asymptotic value.

The relative growth function can also be defined as a decreasing exponential function of time:

$$\frac{1}{Y} \frac{dY}{dt} = k e^{-c e^{-k t}}$$

The integration of Equation (6.10) leads to the yield function:

$$Y = A e^{-c e^{-k t}} \quad (6.11)$$

with $c = (\log A - \log Y_0) e^{k t_0} = \log\left(\frac{A}{Y_0}\right) e^{k t_0}$

4.4 Richards function

Richards (1959) generalized the function presented by von Bertalanffy (1938) for animal growth (Pienaar and Turnbull, 1973). This function describes the absolute growth rate as the difference between an anabolic rate (constructive metabolism), which in plants is proportional to the photosynthetically active area and a catabolic rate (destructive metabolism),

proportional to biomass. If the photosynthetically active area is expressed as an allometric relationship with biomass, these relationships can be expressed as:

Anabolic rate	$c_1 S = c_1 (c_0 Y^m) = c_2 Y^m$
Catabolic rate	$c_3 Y$
Potential growth rate	$c_2 Y^m - c_3 Y$
Growth rate	$c_4 (c_2 Y^m - c_3 Y)$,

where S is the photosynthetically active area; Y is the biomass (or other tree/stand variable); m is the allometric constant of the relationship between S and Y ; c_0, c_1, c_2, c_3 are proportionality coefficients; and c_4 is an efficacy coefficient.

The following differential form of the Richards's function is then obtained:

$$\frac{dY}{dt} = aY^m - bY \quad (6.12)$$

By integration (Bernoulli differential equation), the corresponding yield function is obtained:

$$Y = A \left(1 - ce^{-kt} \right)^{\frac{1}{1-m}}, \quad (6.13)$$

where the parameters m, c, k and A are:

$$c = e^{-(1-m)bt_0} = e^{-kt_0}$$

$$k = (1-m)b$$

$$A = \left(\frac{a}{b} \right)^{\frac{1}{1-m}} \quad (\text{asymptote})$$

In the original version by von Bertalanffy the m exponent was equal to 2/3. It is important to note that the monomolecular, logistic and Gompertz functions are particular cases of the Richards' function when the parameter m takes, respectively, the values 0, 2, or tends to 1.

Figure 6.5 shows the flexibility of the Richard's function as well as the effect of the 3 parameters on the respective shape, while Fig.s 6.6 and 6.7 show its effect on the location of the inflection point and the corresponding Y value.

Higher values of k produce higher growth rates while, on the contrary, smaller values of m result in higher growth rates (Fig. 6.5). As expected, the asymptote is also positively related with higher yields. The combination of the effect of the 3 parameters can, however, “mask” the individual effect of each parameter as is shown in Fig. 6.5 D.

Figure 6.6 shows that higher values of the k parameter result in earlier inflexion points, while the opposite relationship can be observed with the m parameter. The value of Y at the time when the inflexion point occurs is higher for higher asymptote values, but it is inversely related with the value of the m parameter.

Causton and Venus (1981) present a very detailed study on the application of the Richards’s function to plant growth modelling.

Since its introduction to the forest world by Pienaar and Turnbull (1973), this equation has been used more than any other function in studies of tree and stand growth (Zeide, 1993). Adding one parameter to the previous models (6.4.1 to 6.4.3) implies an enhanced flexibility (Fig. 6.5) that allows the fitting to growth data of different shapes. Some authors question the usefulness of the Richard’s function due to its intrinsic properties. Ratkowsky (1983, pp. 83-84) showed that this equation is “the only model that has an unacceptable intrinsic nonlinearity as the solution locus departs significantly from an hyperplane”. These properties may lead, in practice, to instability in the parameter estimates. One way to overcome this problem is to use expert judgement for an estimate for the asymptote and apply nonlinear regression to estimate the remaining parameters from data fitting.

5 Functions of the Hossfeld IV type

5.1 The Hossfeld IV function

The Hossfeld IV function is a sigmoid function, originally proposed in 1822 (Zeide 1993), for the description of tree growth:

$$Y = \frac{t^k}{c+t^k/A} = A \frac{t^k}{Ac+t^k} = A \frac{t^k}{c_1+t^k} \quad (6.14)$$

where A is the asymptote, t is the age and c and k are parameters.

The equation can also be obtained from the generalized logistic in Eq. (6.9) by using $f(\mathbf{X}, t) = -k \ln(t)$. Consequently some authors (e.g. Ciesewski, 2000; Dieguéz-Aranda, 2006) designate it by the log-logistic growth function.

To our knowledge the equation is not based on any specific biological rationales. Despite this fact, this equation performs remarkably well. According to Kiviste (1988), it is the third most accurate of 31 three-parameter equations when the three main stand variables (total tree height, stem diameter and volume) are considered together. Kiviste further found it to be the best equation for volume growth. The Hossfeld IV equation is almost as accurate as the Chapman-Richards equation.

The Hossfeld IV function is able to take several shapes (Fig. 6.8) and to produce inflection points located earlier or later in the life of the tree or stand (Fig.s 6.9 and 6.10).

5.2 McDill-Amateis/Hossfeld IV function

McDill and Amateis (1992) proposed the use of a growth function, written in differential form, whose integral form is equivalent to the Hossfeld IV function. The McDill-Amateis function was developed in order to guarantee the compatibility between dimensions and also taking into account the biological properties expected from growth functions. The variables considered for the growth function and the respective dimensions were:

Variable	dY/dt	t	Y	A
Dimension	LT^{-1}	T	L	L

where L indicates a length, T is a time and A is the asymptote for variable T.

Applying dimensional analysis to these variables (McDill and Amateis, 1992) and taking into account, at the same time, that the growth rate tends to zero when Y tends to the asymptote, the following differential form is obtained:

$$\frac{dY}{dt} = k \frac{Y}{t} \left(1 - \frac{Y}{A} \right) \quad (6.14)$$

In this equation, k is a parameter related to the growth rate. The function has one less parameter than the Richards's function which can be an advantage when fitting the function to empirical data.

The solution for the differential equation leads to the following yield function, known in forestry literature as the McDill-Amateis function:

$$Y = \frac{A}{1 - \left(1 - \frac{A}{Y_0}\right) \left(\frac{t_0}{t}\right)^k} \quad (6.15)$$

where (t_0, Y_0) is the initial condition.

By making $c = \left(\frac{1}{Y_0} - \frac{1}{A}\right) t_0^k$ the integral form of the McDill-Amateis's function coincides with the Hossfeld IV function (Eq. 13). McDill and Amateis's formulation allows a better explanation for the parameters in Eq. (6.13); the k parameter expresses the growth rate and parameter c is related to the initial conditions. The inflexion point occurs when

$$Y = \frac{A}{2} \left(1 - \frac{1}{k}\right)$$

5.3 Generalizations of the Hossfeld IV function

In Kiviste's study the most accurate equations with three (Levakovic III equation) and more (Levakovic I and Yoshida I equations) parameters are modifications of the Hossfeld IV equation:

Levakovic I function:

$$Y = A \left(\frac{t^k}{c_1 + t^k} \right)^{c_2}$$

Levakovic III function:

$$Y = A \left(\frac{t^2}{c_1 + t^2} \right)^{c_2}$$

Yoshida I function:

$$Y = A \frac{t^k}{c_1 + t^k} + c_2$$

6 Other growth functions

Two growth functions that do not fall within any of the previous categories but deserve being mentioned as they have been sometimes used in forest modelling with success (Zeide, 1993; Kiviste et al., 2002), the Weibull and the Sloboda functions.

The Weibull probability distribution function multiplied by a parameter A has been used with success in forest modeling (Yang et al. 1978; Payandeh and Wang 1995):

$$Y = A \left(1 - e^{-kt^b} \right)$$

The Sloboda equation is a generalization of the Gompertz equation by adding a parameter:

$$Y = A e^{-c} e^{-kt^b}$$

7 Zeide decomposition of growth functions

When analysing a large set of growth functions in differential form Zeide (1993) found that all the investigated equations could be decomposed into two components: growth expansion and growth decline. The expansion component represents the innate tendency towards exponential multiplication and is associated with biotic potential, photosynthetic activity, absorption of nutrients, constructive metabolism, anabolism, etc. The decline component represents the constraints imposed by external (competition, limited resources, respiration, and stress) and internal (self-regulatory mechanisms and aging) factors. Those factors that adversely affect growth have been referred to as environmental resistance, destructive metabolism, catabolism, respiration, and so on.

The decomposition can be achieved either by a subtraction or a division – subtraction of logarithms – of the two effects. As was shown before, the Richards's function differential form was defined using these concepts. Analysing the decomposition of the growth functions by division, and rewriting the equations in order to simplify the notation of the constant parameters, Zeide (1993) found that all the equations analyzed, except Weibull's, are particular cases of the two following forms:

$$\ln(y') = k + p \ln(y) + q \ln(t) \quad \leftrightarrow \quad y' = k_1 y^p t^q$$

$$\ln(y') = k + p \ln(y) + q t \quad \leftrightarrow \quad y' = k_1 y^p e^{q t}$$

where $p > 0$, $q < 0$ and $k_1 = e^k$.

In both forms the expansion component is proportional to $\ln(y)$ or, in the antilog form, is a power function of size. The forms differ in the decline component: in the first form, designated by *LTD* form, the decline component is proportional to the logarithm of age while in the second form, the *TD* form, it is either a power function or an exponential function of age. Depending on the values of p and q , several distinct integral equations can be obtained from the same equation form. From the functions listed on Table 6.2, the *LTD* form includes the Lundqvist-Korf's and the McDill-Amateis's (Hossfeld IV) functions and the *TD* form includes the Richards's function and all its particular cases.

The transformed equations reveal quite different and simple relationships between the growth functions. Despite the difference of their integral forms, Hossfeld IV and Korf equations are varieties of the same basic form. On the other hand, differentiation shows that the outward similarity between the Chapman-Richards and Weibull equations is misleading.

Zeide (1993) proposed a third form in which the declining component is expressed as a function of size instead of age, the *YD* form:

$$\ln(y') = k + p \ln(y) + q y \quad \leftrightarrow \quad y' = k_1 y^p e^{q y}$$

The three forms above are very useful for the direct modeling of tree and/or stand growth. Using one of these forms instead of a simple multiple linear regression, provides some assurance that the resulting model will have an appropriate behavior from a biological stand point.

8 Formulating growth functions as a difference equation without t explicit

Age is commonly employed as a variable in growth and yield modeling of even-aged stands. However, age is not always readily available. Increment cores (to the pith) can be used for age determination in tree species with well-defined tree rings, but coring may not be possible for certain species or in tropical zones. In uneven-aged stands, composed of trees that differ markedly in age, stand age cannot be used as a predictor of growth and yield. Hence, an

alternative to the typical applications of growth functions for age determinate trees and stands is sometimes needed.

Tomé et al. (2006) showed that it is possible to obtain formulations of the growth functions as difference equations in which the age is not explicit. Age independent difference forms can be obtained by solving the equation for age (t) and substituting it in the expression of the growth function for age equal to $t+a$. To illustrate the procedure assume the Lundqvist function and solve for age t :

$$Y_t = A e^{-k \frac{1}{t^m}} \Rightarrow t = \left[\frac{-k}{\ln(Y_t/A)} \right]^{\frac{1}{m}}$$

The expression for t can be substituted in the growth function written for age $t+a$, where a is the projection length:

$$Y_{t+a} = A e^{-k \frac{1}{\left(\left[\frac{-k}{\ln(Y_t/A)} \right]^{\frac{1}{m}} + a \right)^m}$$

(Lundqvist function without t explicit)

which results in the formulation of the Lundqvist function as an age independent equation. The expression seems complex but perhaps due to its intrinsic biologic underpinning, parameter estimates converge very well when fitted to data.

Using a similar procedure, the following formulations of the Richards and Hossfeld IV functions as age independent equations can be obtained:

$$Y_{t+a} = A \left(1 - e^{-k a \left(1 - \left(\frac{Y_t}{A} \right)^{1-m} \right)} \right)^{\frac{1}{1-m}}$$

(Richards function without t explicit)

$$Y_{t+a} = A \frac{\left(\left(\frac{c Y_t}{A - Y_t} \right)^{\frac{1}{k}} + a \right)^k}{c + \left(\left(\frac{c Y_t}{A - Y_t} \right)^{\frac{1}{k}} + a \right)^k}$$

(Hossfeld IV function without t explicit)

In order to model growth with these equations, at least one of the parameters has to be expressed as a function of site variables – soil and climate – and stand characteristics. It is important to stress that the age-independent difference equations are invariant for projection length only if the parameters are expressed solely as a function of other variables that are invariant with time. Otherwise, if the parameters are expressed as a function of variables that vary with time, the projections will of course depend on the projection interval used.

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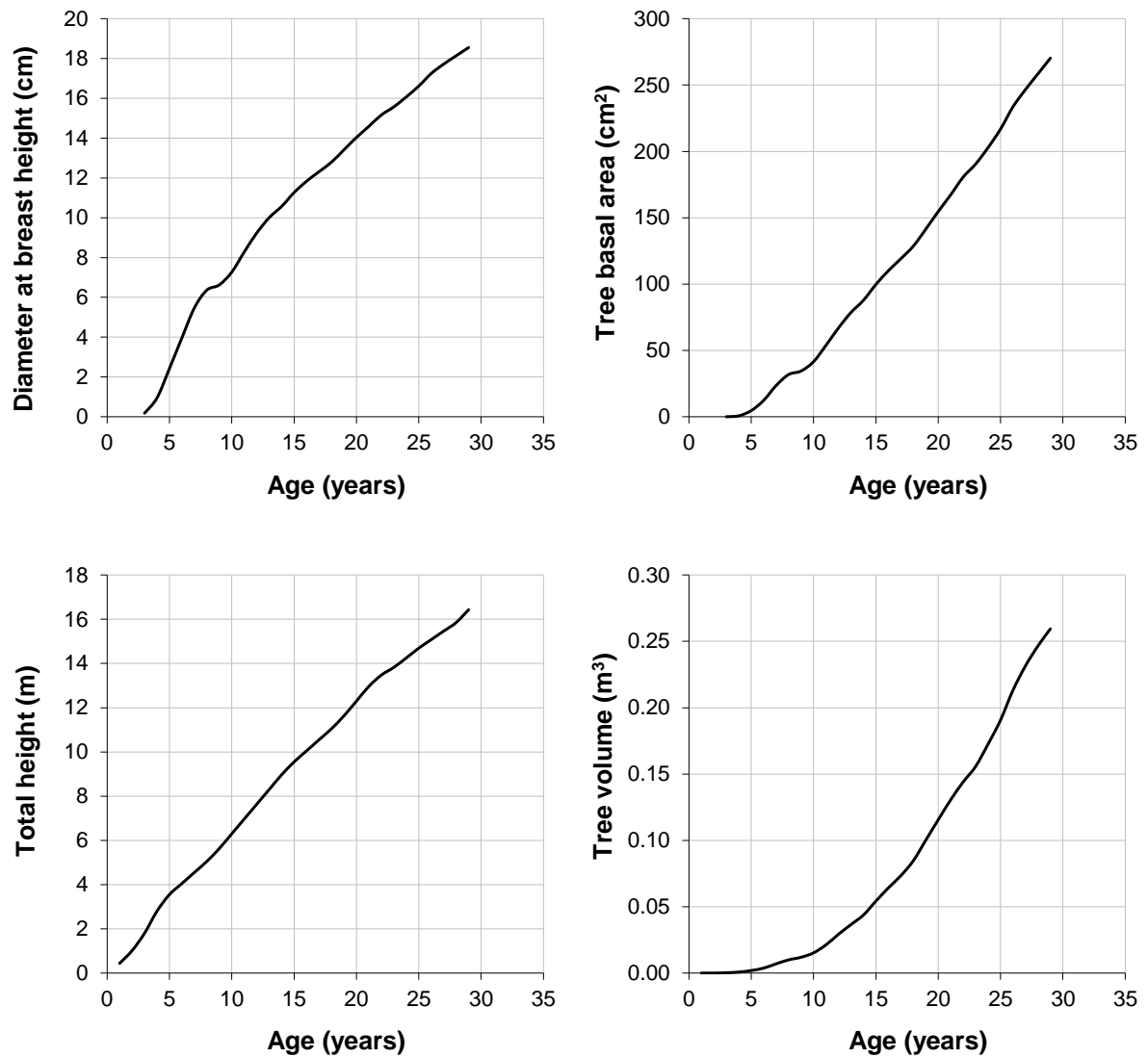


Figure 6.1 Evolution of one tree over time (data from stem analysis of a maritime pine tree at age 29 years).

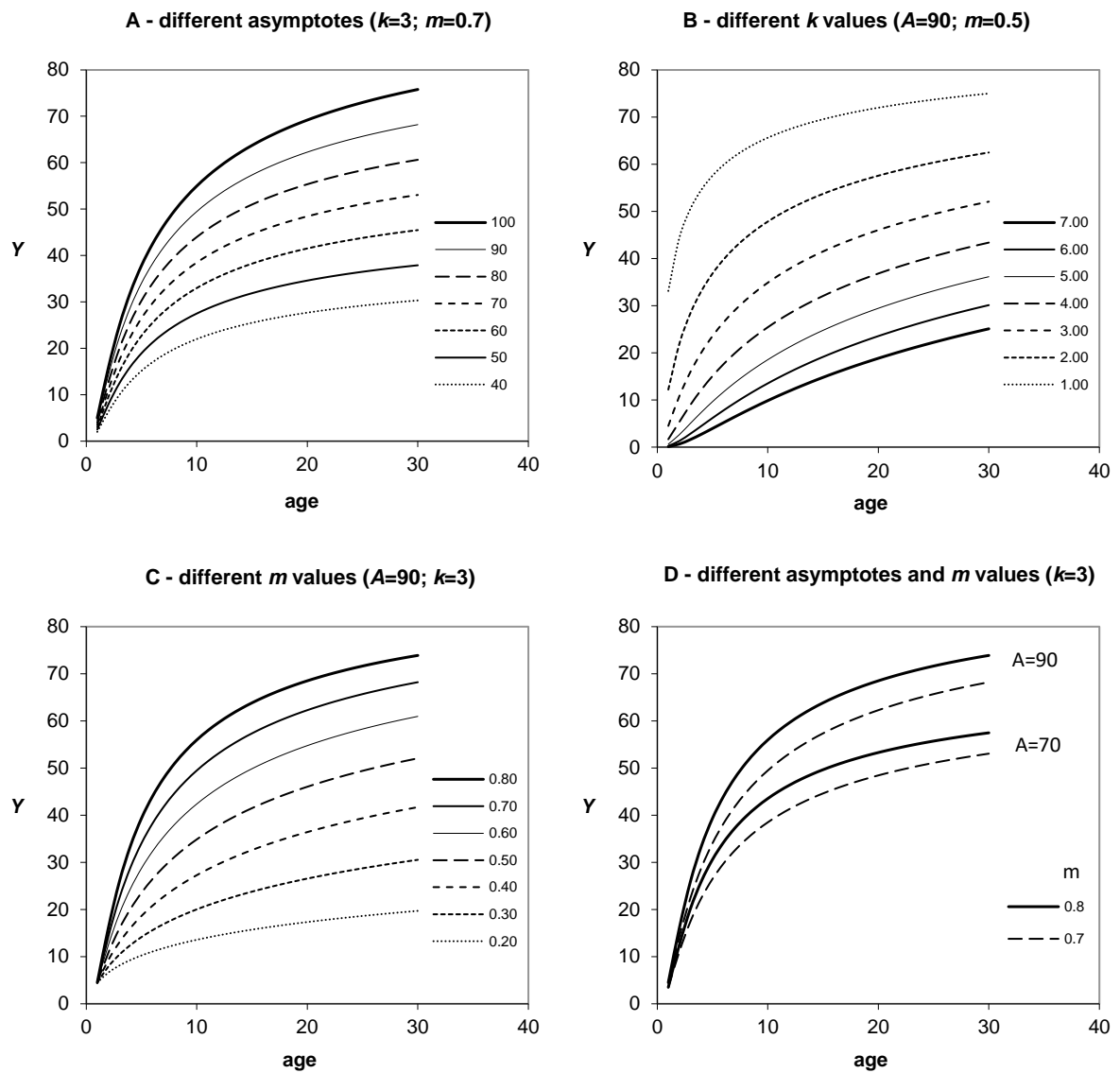


Figure 6.2. Flexibility of the Lundqvist-Korf growth function exhibited through specifying varying values for the parameters.

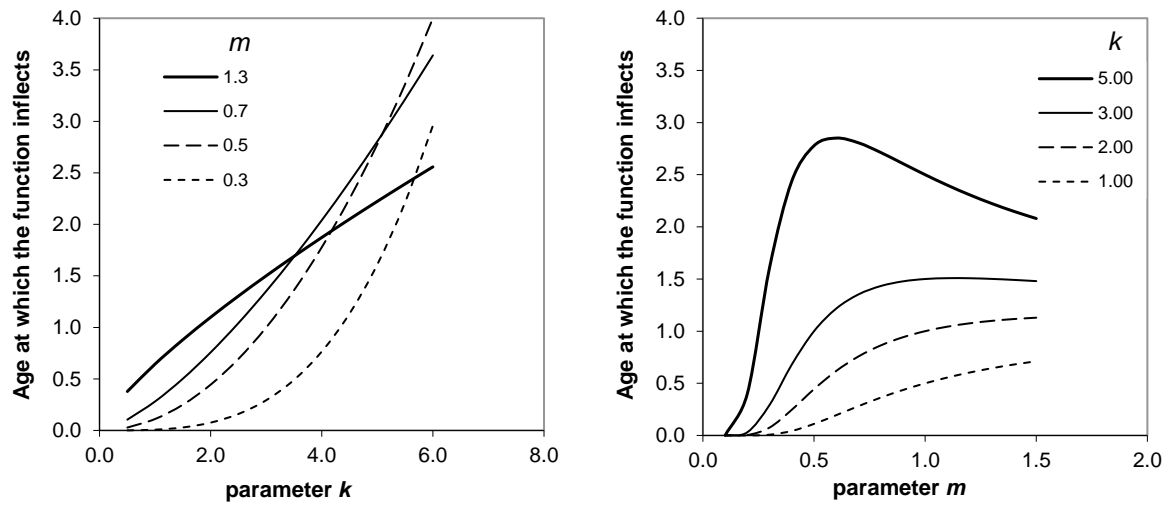


Figure 6.3. Effect of the k and m parameters of the Lundqvist-Korf's function on the location of the inflexion point.

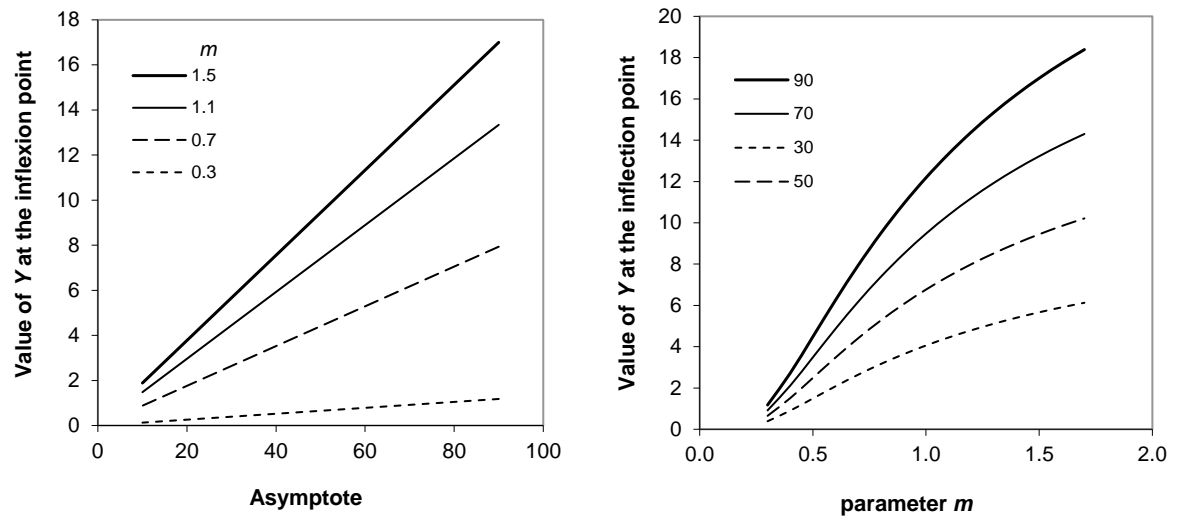


Figure 6.4. Effect of the A and m parameters of the Lundqvist-Korf's function in the Y value at the time when the inflexion point occurs.

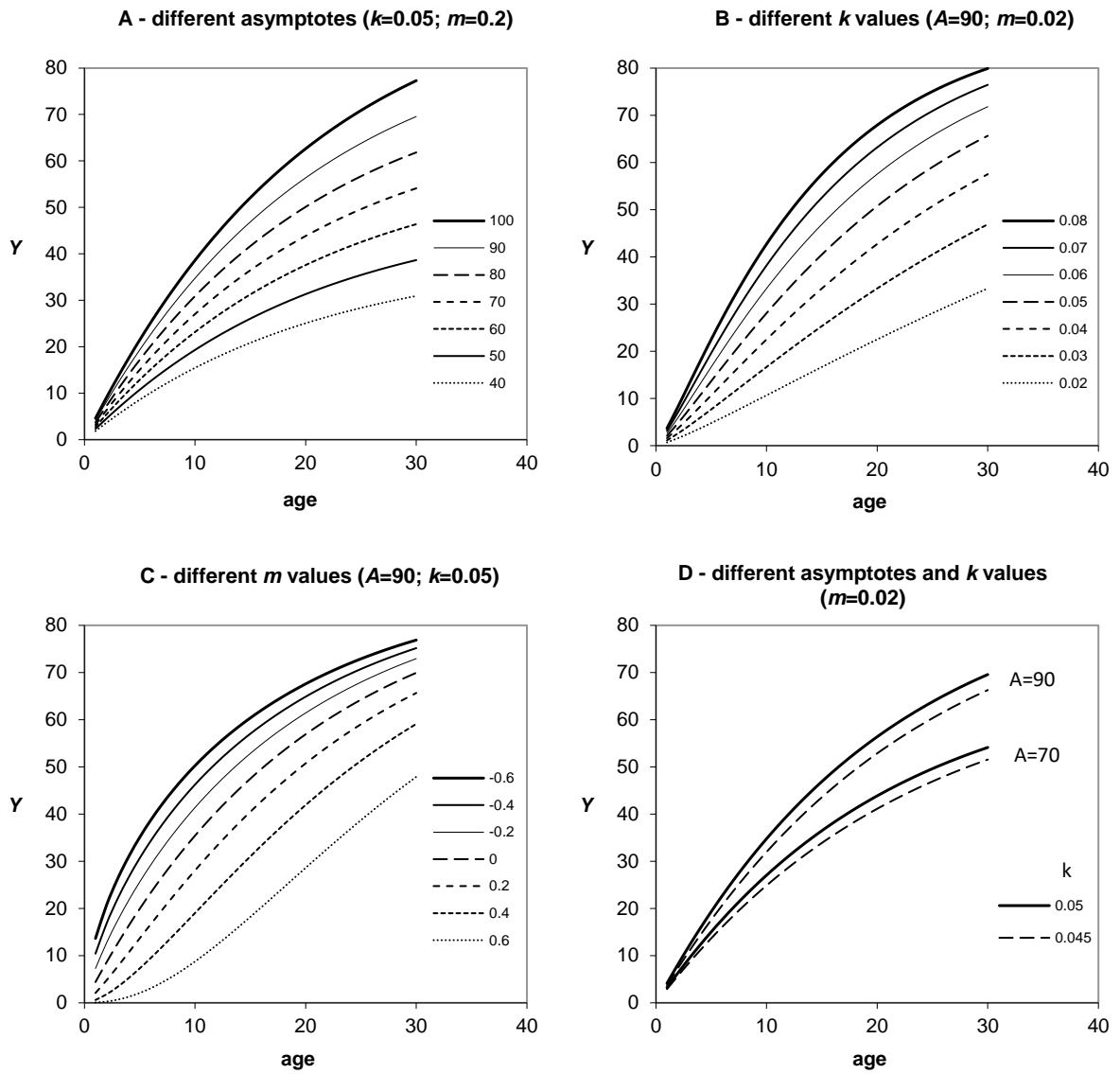


Figure 6.5. Flexibility of the Richards's growth functions with changing values of the parameters.

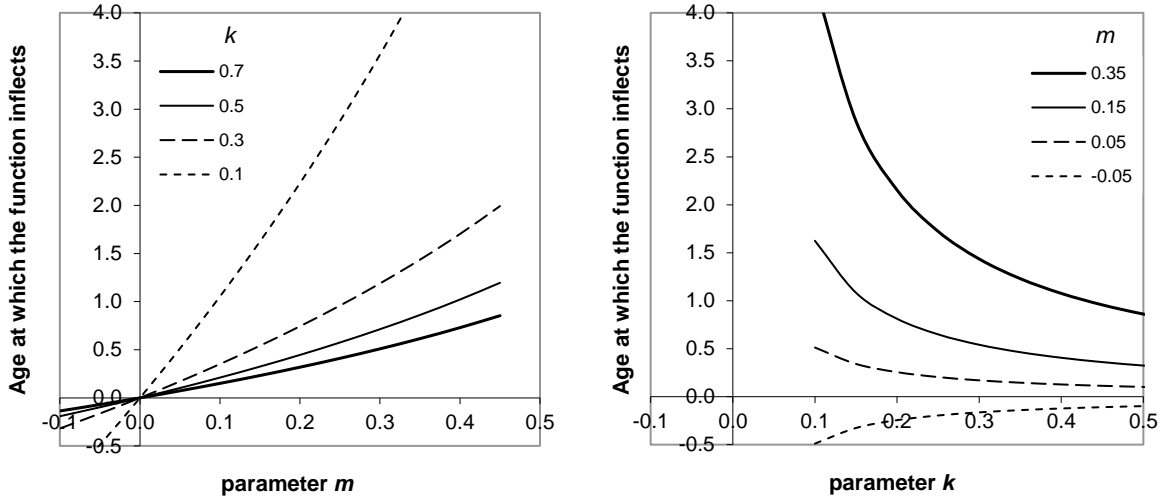


Figure 6.6. Effect of the k and m parameters of the Richards's function on the location of the inflection point.

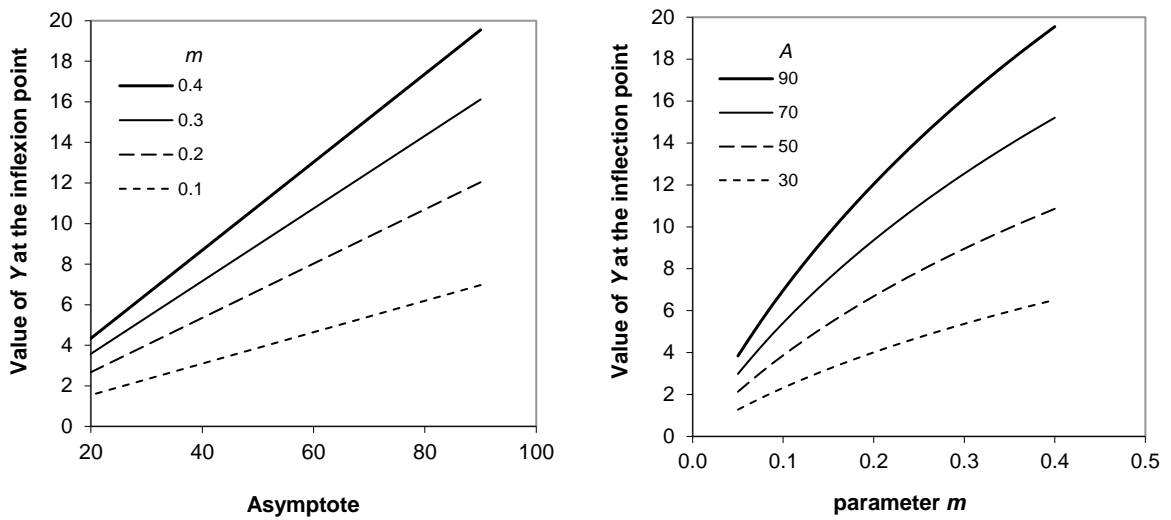


Figure 6.7 Effect of the A and m parameters of the Richards's function in the Y value at the time when the inflection point occurs.

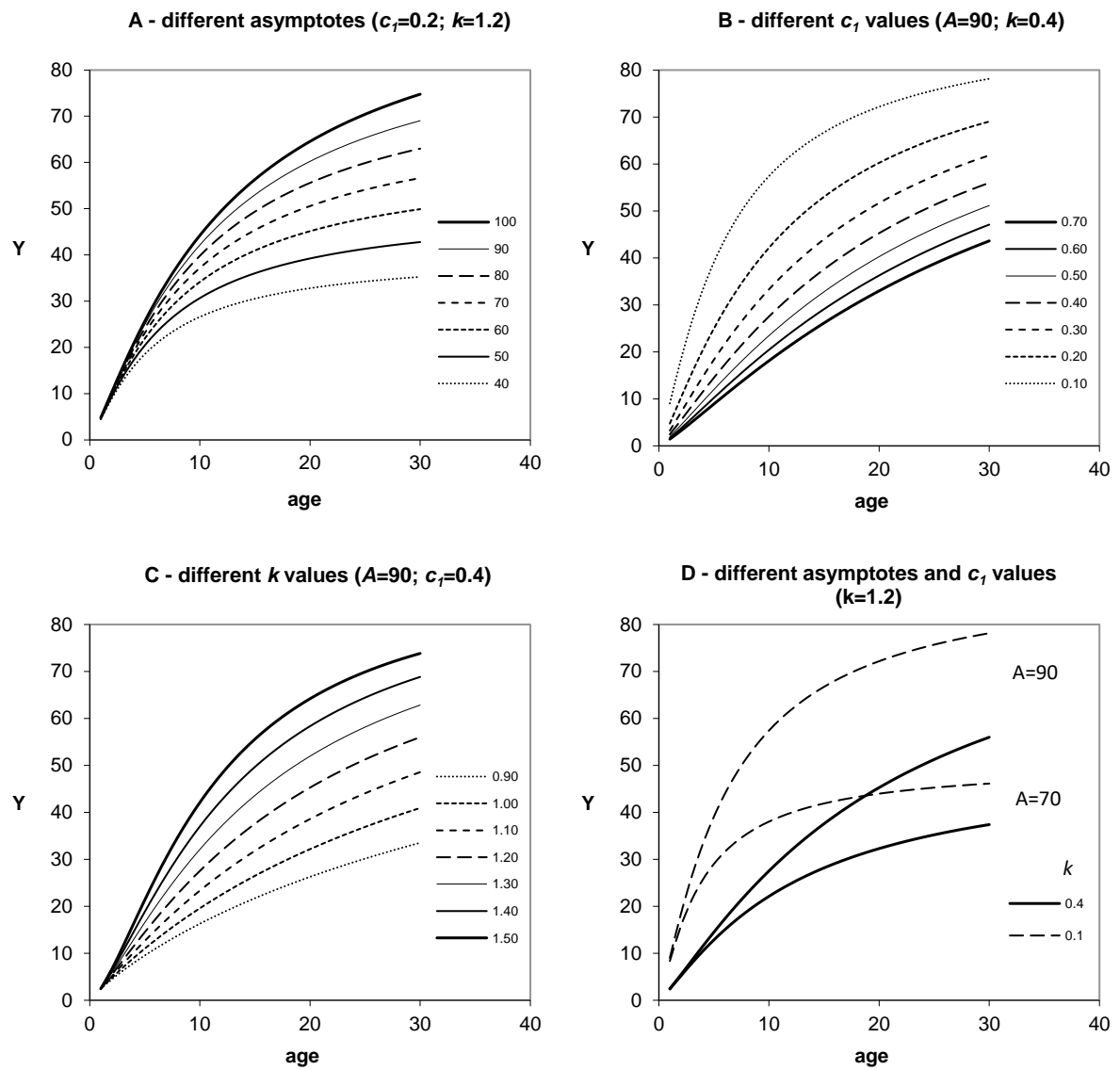


Figure 6.8. Flexibility of the Hossfeld IV growth functions when the parameters take different values.

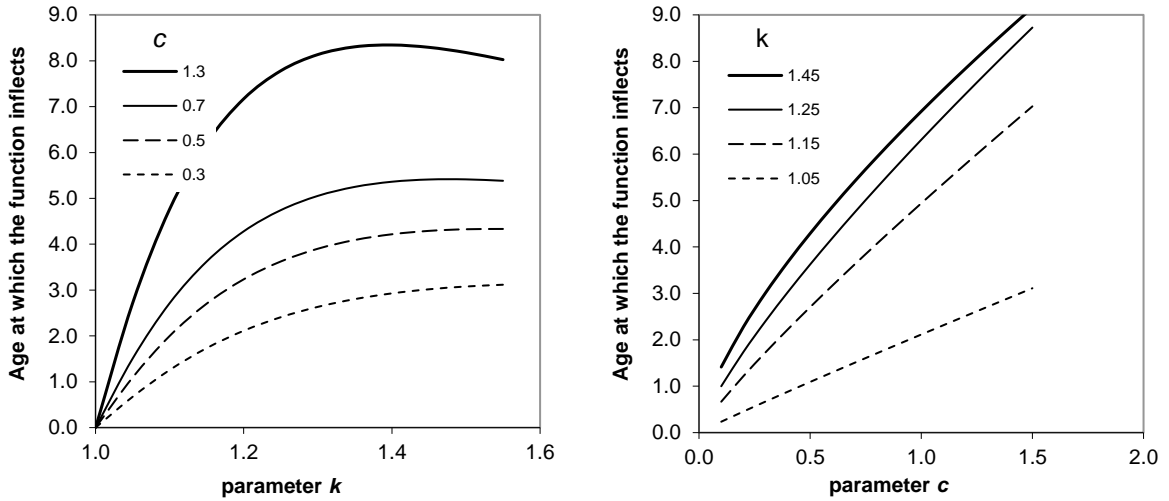


Figure 6.9. Effect of the k and c parameters of the Hossfeld IV function on the location of the inflexion point.

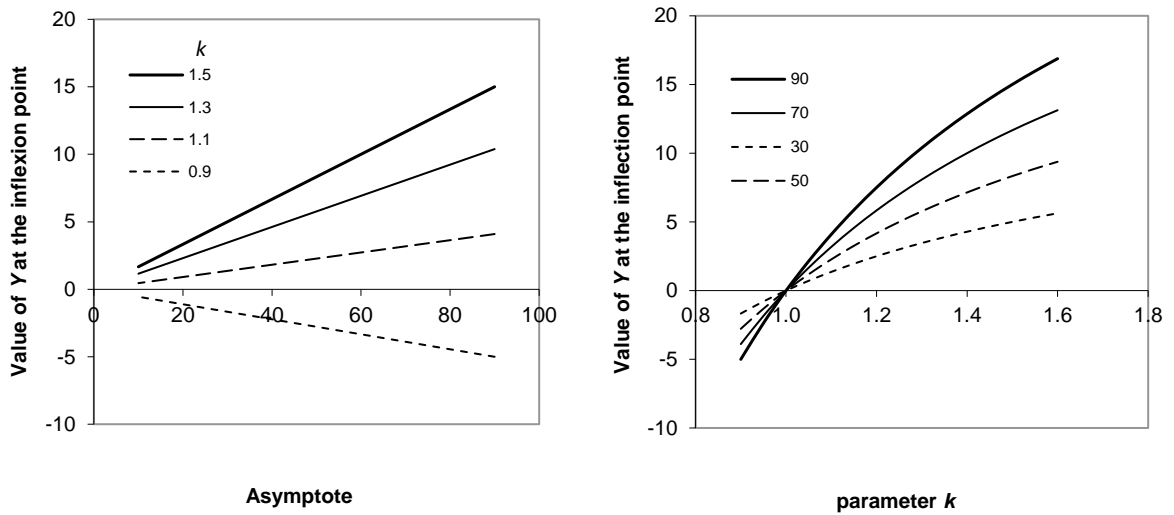


Figure 6.10 Effect of the A and c parameters of the Hossfeld IV function in the Y value at the time when the inflexion point occurs.

Table 6.1. Some non-sigmoid functions commonly used in forest growth modelling

Author or designation	Mathematical expression			Properties		
	Integral form (yield)	Diferencial form (growth)	Restrictions in the parameters	Value at the origin	Inflection point	Asymptote $t \rightarrow \infty$
Hiperbole	$Y = a_0 + a_1 \frac{1}{t}$	$\frac{dY}{dt} = -a_1 \frac{1}{t^2}$	$a_1 < 0$	$t \rightarrow 0; Y \rightarrow -\infty$	None	$Y = a_0$
-	$Y = a_0 + a_1 \frac{1}{t} + a_2 t$	$\frac{dY}{dt} = a_2 - a_1 \frac{1}{t^2}$	$a_1 < 0; a_2 > 0$	$t \rightarrow 0; Y \rightarrow -\infty$	None	$Y \rightarrow \infty$
Linear-logarithmic	$Y = a_0 + a_1 \log t$	$\frac{dY}{dt} = a_1 - \frac{1}{t}$	$a_1 > 0$	$t \rightarrow 0; Y \rightarrow -\infty$	None	$Y \rightarrow \infty$
-	$Y = a_0 + a_1 t^{a_2}$	$\frac{dY}{dt} = \frac{a_2}{t} (Y - a_0)$	$a_1, a_2 > 0; a_2 < 1$ $a_1, a_2 < 0$	$t = 0; Y = a_0$ $t \rightarrow 0; Y \rightarrow -\infty$	None None	$Y \rightarrow \infty$ $Y \rightarrow a_0$
-	$Y = (a_0 + a_1 t)^{a_2}$	$\frac{dY}{dt} = \frac{a_0 a_2 Y}{a_0 + a_1 t}$	$a_1, a_2 > 0; a_2 < 1$	$t = 0; Y = a_0^{a_2}$	None	$Y \rightarrow \infty$
Exponential	$Y = a_0 + a_1 e^{a_2 t}$	$\frac{dY}{dt} = a_2 (a_0 - Y)$	$a_1, a_2 < 0$	$t = 0; Y = a_0 + a_1$	None	$Y \rightarrow a_0$
Freese	$Y = a_0 t^{a_1} + a_2 t$	$\frac{dY}{dt} = Y \left(\log a_2 + \frac{a_1}{t} \right)$	$a_0, a_1 > 0; \ln a_2 < 0$ $a_0, a_1 > 0; \ln a_2 > 0$ $a_0, a_1 > 0; \ln a_2 = 0$	$t = 0; Y = 1$	$t = \frac{-a_1 \pm \sqrt{a_1}}{\ln a_2}$ None	$Y \rightarrow -\infty$
Hossfeld I	$Y = \frac{t^2}{a_0 + a_1 + a_2 t^2}$	$\frac{dY}{dt} = Y^2 \left(\frac{2a_0 + a_1 t}{t^3} \right)$	$a_0 > 0; a_1 < 0$ $a_0 > 0; a_1 > 0$	$t = 0; Y = 0$ $t = 0; Y = 0$	$a_1 a_2 t^3 + 3a_0 a_2 t^2 - a_0^2 = 0$	$Y \rightarrow \frac{1}{a_2}$
Korsun (logarithmic parabole)	$Y = a_0 t^{a_1 + a_2 \log t}$ $a_0 > 0$	$\frac{dY}{dt} = \frac{Y}{t} (a_1 + 2a_2 \log t)$	$a_1 > 0; a_2 < 0$	$t \rightarrow 0; Y \rightarrow 0$	$z^2 - z + 2a_2 = 0$ $z = \beta_1 + 2a_2 \ln t$	$Y \rightarrow 0$

Table 6.2. Theoretical/sigmoid growth functions commonly used in forest growth modelling

Author or designation	Mathematical expression			Properties		
	Integral form (yield)	Diferencial form (growth)	Restrictions in the parameters	Value at the origin	Inflection point	Asymptote $t \rightarrow \infty$
Schumacher	$Y = A e^{-\frac{k}{t}}$	$\frac{dY}{dt} = Y \frac{k}{t^2}$	$k > 0$	$t \rightarrow 0; Y \rightarrow 0$	$t = \frac{k}{2}; Y = \frac{A}{e^2}$	$Y \rightarrow A$
Johnson-Schumacher	$Y = A e^{-\frac{k}{t+a}}$	$\frac{dY}{dt} = Y \frac{k}{(t+a)^2}$	$k > 0$	$t \rightarrow 0; Y \rightarrow Ae^{\frac{k}{a}}$	$t = \frac{k}{2} - a; Y = \frac{A}{e^2}$	$Y \rightarrow A$
Lundqvist-Korf	$Y = Ae^{-\frac{k}{t^m}}$	$\frac{dY}{dt} = mY \frac{k}{t^{m+1}}$	$k > 0$ $n > 0$	$t \rightarrow 0; Y \rightarrow 0$	$t = \left(\frac{mk}{m+1}\right)^{\frac{1}{m}}; Y = Ae^{-\frac{m+1}{m}}$	$Y \rightarrow A$
Monomolecular	$Y = A (1 - ce^{-kt})$	$\frac{dY}{dt} = k(A - Y)$	$k > 0$	$t = 0; Y = A(1 - c)$	None	$Y \rightarrow A$
Logística	$Y = \frac{A}{(1 + ce^{-kt})}$	$\frac{dY}{dt} = \frac{k}{A}(AY - Y^2)$	$k > 0$	$t = 0; Y = \frac{A}{1+c}$ $t \rightarrow -\infty; Y \rightarrow 0$	$t = \frac{1}{k} \log c; Y = \frac{A}{2}$	$Y \rightarrow A$
Pearl-Reed ¹	$Y = \frac{A}{1 + ce^{-f(\mathbf{X}, t)}}$	$\frac{dY}{dt} = \frac{AY - Y^2}{A} f(\mathbf{X}, t)$	$c > 0$	$t = 0; Y = \frac{A}{1+c}$ $t \rightarrow -\infty; Y \rightarrow 0$	Inflection depends on $f(\mathbf{X}, t)$	$Y \rightarrow A$
Gompertz	$Y = A e^{-ce^{-kt}}$	$\frac{dY}{dt} = k Y \ln \frac{A}{Y}$	$k > 0$ $c > 0$	$t = 0; Y = A e^{-c}$ $t \rightarrow -\infty; Y = 0$	$t = \frac{\log c}{k}; Y = \frac{A}{e}$	$Y \rightarrow A$
Richards	$Y = A \left(1 - ce^{-kt}\right)^{\frac{1}{1-m}}$	$\frac{dY}{dt} = \frac{kY}{1-m} \left[\left(\frac{A}{Y}\right)^{1-m} - 1\right]$	$k > 0$	$t = 0; Y = A (1 - c)^{\frac{1}{1-m}}$	$t = \frac{\log\left(\frac{c}{1-m}\right)}{k}; Y = Am^{\frac{1}{1-m}}$	$Y \rightarrow A$
McDill-Amateis / Hossfeld / log-logistic	$Y = A \frac{t^k}{c + t^k}$	$\frac{dY}{dt} = k \frac{Y}{t} \left(1 - \frac{Y}{A}\right)$	$k > 1$	$t \rightarrow 0; Y \rightarrow 0$	$t = \left[\frac{c(k-1)}{k+1}\right]^{1/k}; Y = \frac{A}{2} \left(1 - \frac{1}{k}\right)$	$Y \rightarrow A$

¹depending on the *function* $f(\mathbf{X}, t)$ may not be a sigmoidal function

