Uma resolução do 1º Teste 2021/22 (1h30m)

(5.5v) **1.** Considere $A = \begin{bmatrix} -2 & -6 & 0 \\ \alpha & 1 & 2 \\ 0 & -1 & \alpha \\ 1 & 3 & 0 \end{bmatrix} = [u_1 \quad u_2 \quad u_3]$ e $b = (2,-1,0,\beta)$, com $\alpha, \beta \in \mathbb{R}$.

a) Discuta o sistema $Ax = b$ para todos os valores de $\alpha, \beta \in \mathbb{R}$.

$$\left[\begin{array}{ccc|c} -2 & -6 & 0 & 2 \\ \alpha & 1 & 2 & -1 \\ 0 & -1 & \alpha & 0 \\ 1 & 3 & 0 & \beta \end{array}\right] \xrightarrow[L_4+\frac{1}{2}L_1 \to L_4]{L_2+\frac{\alpha}{2}L_1 \to L_2} \left[\begin{array}{ccc|c} -2 & -6 & 0 & 2 \\ 0 & -3\alpha+1 & 2 & \alpha-1 \\ 0 & -1 & \alpha & 0 \\ 0 & 0 & 0 & 1+\beta \end{array}\right] \xrightarrow[L_2 \leftrightarrow L_3]{}$$

$$\to \left[\begin{array}{ccc|c} -2 & -6 & 0 & 2 \\ 0 & -1 & \alpha & 0 \\ 0 & -3\alpha+1 & 2 & \alpha-1 \\ 0 & 0 & 0 & 1+\beta \end{array}\right] \xrightarrow[L_3+(-3\alpha+1)L_2 \to L_3]{} \left[\begin{array}{ccc|c} -2 & -6 & 0 & 2 \\ 0 & -1 & \alpha & 0 \\ 0 & 0 & (-3\alpha+1)\alpha+2 & \alpha-1 \\ 0 & 0 & 0 & 1+\beta \end{array}\right] = [A' \mid b']$$

Se $1+\beta \neq 0 \Leftrightarrow \beta \neq -1, \forall\, \alpha \in \mathbb{R}$, **S.IMP.**

$(-3\alpha+1)\alpha + 2 = 0 \Leftrightarrow -3\alpha^2 + \alpha + 2 = 0 \Leftrightarrow \alpha = \frac{-1 \pm \sqrt{1+24}}{-6} = \frac{-1\pm5}{-6} \Leftrightarrow \alpha = -\frac{2}{3} \vee \alpha = 1$

Se $\beta = -1$

e $\alpha \neq -\frac{2}{3} \wedge \alpha \neq 1$, **S.P.D.**

e $\alpha = -\frac{2}{3}$, $[A' \mid b'] = \left[\begin{array}{ccc|c} -2 & -6 & 0 & 2 \\ 0 & -1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$, **S.IMP.**

e $\alpha = 1$, $[A' \mid b'] = \left[\begin{array}{ccc|c} -2 & -6 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, **S.P.IND.**

b) Indique os valores de α , β para os quais:

i) $(-4,1,1)$ é solução de $Ax = b$.

$$\begin{bmatrix} -2 & -6 & 0 \\ \alpha & 1 & 2 \\ 0 & -1 & \alpha \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4\alpha+3 \\ -1+\alpha \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ \beta \end{bmatrix} \Leftrightarrow \begin{cases} 2 = 2 \\ -4\alpha+3 = -1 \\ -1+\alpha = 0 \\ \beta = -1 \end{cases} \Leftrightarrow \begin{cases} 2 = 2 \\ -1 = -1 \\ \alpha = 1 \\ \beta = -1 \end{cases}$$

$(-4,1,1)$ é solução de $Ax = b$ quando $\alpha = 1$ e $\beta = -1$.

ii) $[u_1 \quad u_2 \quad u_3 \quad b]$ é invertível.

$[u_1 \quad u_2 \quad u_3 \quad b]$ é invertível sse $car([u_1 \quad u_2 \quad u_3 \quad b]) = 4$ (ou sse na matriz $[A' \quad b']$ existem 4 colunas com pivot)

Por a) quando $\beta \neq -1 \wedge \alpha \neq -\frac{2}{3} \wedge \alpha \neq 1$

No que se segue considere $\alpha = 1$ e $\beta = -1$.

c) Indique uma base e a dimensão de $\langle u_1, u_2, u_3, b \rangle$.

De a) tem-se para $\alpha = 1$ e $\beta = -1$, $[u_1 \quad u_2 \quad u_3 \quad b] \to \cdots \to \begin{bmatrix} -2 & -6 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ portanto,

uma base de $\langle u_1, u_2, u_3, b \rangle = \mathcal{C}\left(\begin{bmatrix} -2 & -6 & 0 & 2 \\ 1 & 1 & 2 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 0 & -1 \end{bmatrix}\right)$: $\{(-2,1,0,1), (-6,1,-1,3)\}$ e $\dim\langle u_1, u_2, u_3, b \rangle = 2$

d) Escreva u_3 como combinação linear de u_1 e u_2. O conjunto $\{u_1, u_2, u_3\}$ é linearmente independente?

De a) para $\alpha = 1$ e $\beta = -1$ obtém-se

$$A \to \cdots \to \left[\begin{array}{cc|c} -2 & -6 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[(-1)\times L_2 \to L_2]{} \left[\begin{array}{cc|c} -2 & -6 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[L_1+6L_2 \to L_1]{} \left[\begin{array}{cc|c} -2 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}L_1 \to L_1]{}$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$u_3 = 3u_1 - u_2$ logo $\{u_1, u_2, u_3\}$ não é linearmente independente.

NOTA: efetivamente, $(0,2,1,0) = 3 \times (-2,1,0,1) - (-6,1,-1,3)$

e) Indique uma base de $\mathbb{R}^3$ que inclua o vetor $(-4,1,1)$.

As bases de $\mathbb{R}^3$ são conjuntos linearmente independentes com 3 vetores de $\mathbb{R}^3$.

$\{(-4,1,1), (1,0,0)\}$ é linearmente independente, isto é, $\nexists \lambda \in \mathbb{R}: (-4,1,1) = \lambda(1,0,0)$

$$\begin{bmatrix} -4 & 1 & x_1 \\ 1 & 0 & x_2 \\ 1 & 0 & x_3 \end{bmatrix} \xrightarrow[L_1 \leftrightarrow L_2]{} \begin{bmatrix} 1 & 0 & x_2 \\ -4 & 1 & x_1 \\ 1 & 0 & x_3 \end{bmatrix} \xrightarrow[\substack{L_2+4L_1 \to L_2 \\ L_3+(-1)\times L_1 \to L_3}]{} \begin{bmatrix} 1 & 0 & x_2 \\ 0 & 1 & 4x_2 + x_1 \\ 0 & 0 & -x_2 + x_3 \end{bmatrix}$$

$\{(-4,1,1), (1,0,0), x\}$ é linearmente independente quando $-x_2 + x_3 \neq 0 \Leftrightarrow x_2 \neq x_3, \forall x_1, x_3 \in \mathbb{R}$.

Uma base de $\mathbb{R}^3$, por exemplo: $\{(-4,1,1), (1,0,0), (0,1,0)\}$

(3.0v) 2. Considere $A = \begin{bmatrix} 1 & 2 & -2 & -1 \\ 2 & 1 & -1 & -2 \\ -1 & 1 & -1 & 1 \end{bmatrix}$.

a) Descreva $\mathcal{C}(A)$ analitica e geometricamente.

$\mathcal{C}(A) = \{b \in \mathbb{R}^3 : Ax = b \text{ é um sistema possível}\}$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -1 & b_1 \\ 2 & 1 & -1 & -2 & b_2 \\ -1 & 1 & -1 & 1 & b_3 \end{array}\right] \xrightarrow[\substack{L_2+(-2)\times L_1 \to L_2 \\ L_3+L_1 \to L_3}]{} \left[\begin{array}{cccc|c} 1 & 2 & -2 & -1 & b_1 \\ 0 & -3 & 3 & 0 & -2b_1 + b_2 \\ 0 & 3 & -3 & 0 & b_1 + b_3 \end{array}\right] \xrightarrow[L_3+L_2 \to L_3]{}$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & -2 & -1 & b_1 \\ 0 & -3 & 3 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

$$\mathcal{C}(A) = \left\{(b_1, b_2, b_3) \in \mathbb{R}^3 : -b_1 + b_2 + b_3 = 0\right\}$$

Geometricamente $\mathcal{C}(A)$ é o plano de $\mathbb{R}^3$ que passa na origem e é ortogonal ao vetor $(-1,1,1)$

b) Justifique que $\mathcal{N}(A) = \langle (1,1,1,1), (-1,-2,-2,-1) \rangle$.

Por a) $A \to \cdots \to \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{3}L_3 \to L_3]{} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[L_1+(-2)\times L_2 \to L_1]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\mathcal{N}(A) = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_4, x_2 = x_3, \forall x_3, x_4 \in \mathbb{R}\right\}$$

$\dim \mathcal{N}(A) = 2$ porque as soluções do sistema homogéneo $Ax = 0$ têm 2 variáveis livres – as bases de $\mathcal{N}(A)$ são conjuntos com 2 vetores.

$\{(1,1,1,1), (-1,-2,-2,-1)\}$ é linearmente independente, isto é, $\nexists \lambda \in \mathbb{R}: (-1,-2,-2,-1) = \lambda(1,1,1,1)$

$(1,1,1,1) \in \mathcal{N}(A)$ e $(-1,-2,-2,-1) \in \mathcal{N}(A)$ porque em ambos os vetores $x_1 = x_4$ e $x_2 = x_3$

Qualquer conjunto linearmente independente com 2 vetores de $\mathcal{N}(A)$ é uma base de $\mathcal{N}(A)$. Logo, $\langle (1,1,1,1), (-1,-2,-2,-1) \rangle = \mathcal{N}(A)$.

c) Determine um vetor unitário que seja ortogonal a $\mathcal{N}(A)$.

Por b) $\mathcal{N}(A) = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_4, x_2 = x_3, \forall x_3, x_4 \in \mathbb{R}\right\}$

Uma base $\mathcal{N}(A)$: $\{(0,1,\mathbf{1},\mathbf{0}), (1,0,\mathbf{0},\mathbf{1})\}$

$x \perp \mathcal{N}(A)$ sse $x \perp (0,1,1,0)$ e $x \perp (1,0,0,1)$

$x \perp (0,1,1,0)$ e $x \perp (1,0,0,1) \Leftrightarrow \begin{cases} x.(0,1,1,0) = 0 \\ x.(1,0,0,1) = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = -x_3 \\ x_1 = -x_4 \end{cases}, \forall x_3, x_4 \in \mathbb{R}$

$x \perp (0,1,1,0)$ e $x \perp (1,0,0,1) \Leftrightarrow x = \begin{bmatrix} -x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}, \forall x_3, x_4 \in \mathbb{R}$

$\|x\| = 1 \Leftrightarrow \sqrt{{x_4}^2 + {x_3}^2 + {x_3}^2 + {x_4}^2} = 1 \Leftrightarrow 2{x_4}^2 + 2{x_3}^2 = 1 \Leftrightarrow {x_4}^2 + {x_3}^2 = \frac{1}{2} \Leftrightarrow {x_4}^2 = -{x_3}^2 + \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow x_4 = \pm\sqrt{-{x_3}^2 + \frac{1}{2}}$. Para, por exemplo, $x_3 = 0$, obtém-se $x_4 = \pm\frac{\sqrt{2}}{2}$. Neste caso, para $x_4 = \frac{\sqrt{2}}{2}$, obtém-se $x = \left(-\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right)$

(1.5v) 3.a) Defina subespaço vetorial.

Subespaço vetorial é um subconjunto de $\mathbb{R}^n$ diferente do vazio, fechado para a adição e para o produto por um escalar.

b) Considere a matriz $A = [v_1 \quad v_2 \quad v_3 \quad v_4]$ tal que $\mathcal{N}(A) = \langle(1,0,2,1), (2,0,1,0)\rangle$. Determine uma base de de $\mathcal{C}(A)$.

$\{(1,0,2,1), (2,0,1,0)\}$ é l.i. logo $\dim \mathcal{N}(A) = 2 \Rightarrow \dim \mathcal{C}(A) = 2$

$(2,0,1,0) \in \mathcal{N}(A) \Leftrightarrow (2,0,1,0)$ é solução de $[v_1 \quad v_2 \quad v_3 \quad v_4]x = \vec{0} \Leftrightarrow 2v_1 + 0v_2 + 1v_3 + 0v_4 = \vec{0} \Leftrightarrow$
$\Leftrightarrow v_3 = -2v_1$, isto é, v_3 é combinação linear de v_1.

Logo $\mathcal{C}(A) = \langle v_1, v_2, v_3, v_4\rangle = \langle v_1, v_2, v_4\rangle$

$(1,0,2,1) \in \mathcal{N}(A) \Leftrightarrow (1,0,2,1)$ é solução de $[v_1 \quad v_2 \quad v_3 \quad v_4]x = \vec{0} \Leftrightarrow 1v_1 + 0v_2 + 2v_3 + 1v_4 = \vec{0} \Leftrightarrow$
$\Leftrightarrow v_4 = -v_1 - 2v_3 \Leftrightarrow v_4 = -v_1 - 2 \times (-2v_1) \Leftrightarrow v_4 = 3v_1$, isto é, v_4 é combinação linear de v_1.

Logo $\mathcal{C}(A) = \langle v_1, v_2, v_3, v_4\rangle = \langle v_1, v_2, v_4\rangle = \langle v_1, v_2\rangle$. E como $\dim \mathcal{C}(A) = 2$, então $\{v_1, v_2\}$ é linearmente independente. Portanto, $\{v_1, v_2\}$ é uma base de $\mathcal{C}(A)$.

Algumas alternativas de resolução

1.e) Indique uma base de $\mathbb{R}^3$ que inclua o vetor $(-4,1,1)$.

As bases de $\mathbb{R}^3$ são conjuntos linearmente independentes com 3 vetores de $\mathbb{R}^3$.

$$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[L_1 \leftrightarrow L_2]{} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{L_2+4L_1 \to L_2 \\ L_3+(-1)\times L_1 \to L_3}]{} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$\{(-4,1,1),(1,0,0),(0,1,0)\}$ é linearmente independente porque na matriz em escada correspondente, as primeiras 3 colunas têm pivot.

Uma base de $\mathbb{R}^3$, por exemplo: $\{(-4,1,1),(1,0,0),(0,1,0)\}$

2.b) Justifique que $\mathcal{N}(A) = \langle (1,1,1,1), (-1,-2,-2,-1) \rangle$.

Resolução 2

Por a) $A \to \cdots \to \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{3}L_3 \to L_3]{} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[L_1+(-2)\times L_2 \to L_1]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\mathcal{N}(A) = \left\{ (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 = x_4, x_2 = x_3, \forall x_3, x_4 \in \mathbb{R} \right\}$$

$$\langle (1,1,1,1),(-1,-2,-2,-1) \rangle = \mathcal{C}\left(\begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \right) = \left\{ b \in \mathbb{R}^4 : \begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} x = b \text{ é um sistema possível} \right\}$$

$$\left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 1 & -2 & b_2 \\ 1 & -2 & b_3 \\ 1 & -1 & b_4 \end{array}\right] \xrightarrow[\substack{L_2+(-1)\times L_1 \to L_2 \\ L_3+(-1)\times L_1 \to L_3 \\ L_4+(-1)\times L_1 \to L_4}]{} \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & -1 & -b_1+b_2 \\ 0 & -1 & -b_1+b_3 \\ 0 & 0 & -b_1+b_4 \end{array}\right] \xrightarrow[L_3+(-1)\times L_2 \to L_3]{} \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & -1 & -b_1+b_2 \\ 0 & 0 & -b_2+b_3 \\ 0 & 0 & -b_1+b_4 \end{array}\right]$$

$$\langle (1,1,1,1),(-1,-2,-2,-1) \rangle = \left\{ (b_1,b_2,b_3,b_4) \in \mathbb{R}^4 : -b_2+b_3 = 0, -b_1+b_4 = 0 \right\} =$$

$$= \left\{ (b_1,b_2,b_3,b_4) \in \mathbb{R}^4 : b_1 = b_4, b_2 = b_3, \forall b_3, b_4 \in \mathbb{R} \right\}$$

Portanto $\mathcal{N}(A) = \langle (1,1,1,1),(-1,-2,-2,-1) \rangle$

Resolução 3

Por a) $A \to \cdots \to \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{3}L_3 \to L_3]{} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[L_1+(-2)\times L_2 \to L_1]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\mathcal{N}(A) = \left\{ (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 = x_4, x_2 = x_3, \forall x_3, x_4 \in \mathbb{R} \right\}$$

Uma base de $\mathcal{N}(A)$: $\{(0,1,\mathbf{1},\mathbf{0}),(1,0,\mathbf{0},\mathbf{1})\}$

$\langle (0,1,1,0),(1,0,0,1) \rangle = \langle (1,1,1,1),(-1,-2,-2,-1) \rangle$ **?**

Se $(0,1,1,0)$ e $(1,0,0,1)$ se podem escrever como combinação linear de $(1,1,1,1)$ e $(-1,-2,-2,-1)$ então $\langle (0,1,1,0),(1,0,0,1) \rangle \subseteq \langle (1,1,1,1),(-1,-2,-2,-1) \rangle$

$$\left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[\substack{L_2+(-1)\times L_1 \to L_2 \\ L_3+(-1)\times L_1 \to L_3 \\ L_4+(-1)\times L_1 \to L_4}]{} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[L_3+(-1)\times L_2 \to L_3]{} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Se $(1,1,1,1)$ e $(-1,-2,-2,-1)$ se podem escrever como combinação linear de $(0,1,1,0)$ e $(1,0,0,1)$ então $\langle(1,1,1,1),(-1,-2,-2,-1)\rangle \subseteq \langle(0,1,1,0),(1,0,0,1)\rangle$

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow[L_1 \leftrightarrow L_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow[L_3+(-1)\times L_1 \to L_3]{} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow[L_4+(-1)\times L_2 \to L_4]{}$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\langle(0,1,1,0),(1,0,0,1)\rangle \subseteq \langle(1,1,1,1),(-1,-2,-2,-1)\rangle$ e

$$\langle(1,1,1,1),(-1,-2,-2,-1)\rangle \subseteq \langle(0,1,1,0),(1,0,0,1)\rangle \Leftrightarrow$$

$$\Leftrightarrow \langle(1,1,1,1),(-1,-2,-2,-1)\rangle = \langle(0,1,1,0),(1,0,0,1)\rangle = \mathcal{N}(A)$$