CHAPTER 14: A HARVEST SCHEDULING MODEL WITH A LONGER PLANNING HORIZON

In Chapter 12, two basic rules of thumb for selecting the length of the planning horizon for a harvest scheduling linear program were discussed: 1) the planning horizon should be between one and two rotations long, and 2) the planning horizon is too short if adding one more planning period significantly changes the plan for the first planning period. The linear programming harvest scheduling examples considered so far in this text have had twenty-year planning horizons. By the first rule of thumb, the planning horizons in our initial examples were too short: the optimal rotation for site class II is thirty years, and the optimal rotation for site class I is forty years. This chapter discusses how the planning horizon can be extended. Specifically, the planning horizon will be extended to forty years in this chapter. Forty years is really just the minimal planning horizon for our example – fifty, sixty, or more years would probably be better. However, once you understand how to extend the planning horizon to forty years, you should be able to generalize this knowledge in order to make the planning horizon any length you wish.

The key change that must be made in the formulation in order to accommodate longer planning horizons is to allow areas to be harvested more than once during the planning horizon. This is accomplished by modifying the way the decision variables are defined. The new definition of the variables is presented first; then the effect of this change on the objective function and each type of constraint is discussed. The profit-maximization problem will be used as the primary example. However, the modifications for the cost-minimization problem would be similar.

1. Redefining the Decision Variables to Allow Two Harvests

Recall that the initial example had a twenty-year planning horizon and allowed acres to be harvested only once in the planning horizon. In that model, there were only three possible prescriptions for a given area: harvest in period one, harvest in period 2, or do not harvest during the planning horizon. Without allowing an area to be harvested more than once, if the planning horizon is increased to 40 years – i.e., to four periods – there would be five possible prescriptions: harvest in one of the four periods or do not harvest. However, allowing only one harvest within a 40-year planning horizon means that stands harvested in the first period (in year 5, on average) cannot be harvested again for at least 35 years. This is too long when the optimal rotation for some stands is only 30 years.

When two harvests are allowed within the planning horizon, the number of possible prescriptions increases fairly dramatically. Consider just the prescriptions starting with a harvest in period 1. One could harvest in period 1 only, or in periods 1 and 2, periods 1 and 3, or in periods 1 and 4 – four prescriptions involving a harvest in period 1. We will have three prescriptions starting with a harvest in period 2: harvest in period 2 only, in periods 2 and 3, or in periods 2 and 4. Similarly, we will have two prescriptions with an initial harvest in period 3 and one prescription with a harvest only in period 4. In addition, there will be one prescription involving no harvest at all. This results in a total of eleven prescriptions. Table 14.1 summarizes the possible prescriptions when two harvests are allowed within a 40-year planning horizon. (How many prescriptions would be possible with two possible harvests within a 50-year planning horizon?)

Clearly, adding planning periods can significantly increase the size of the linear programming problem. It is clearly necessary to consider some strategies for reducing the number of possible prescriptions. A good start would be to eliminate prescriptions that involve harvests that are too close together. If we require at least 20 years to pass between harvests, we can eliminate all the prescriptions with harvests in consecutive periods. This eliminates three of the possible prescriptions in Table 14.1 and reduces the number of possible prescriptions for the 40-year planning horizon to eight. Table 14.2 lists the eight possible prescriptions with up to two harvests within a 40-year planning horizon and a

Table 14.1. Harvest scheduling prescriptions with up to two possible harvests within a 40-year planning horizon.

Prescription	Planning Period			
	1	2	3	4
1	Harvest			
2	Harvest	Harvest		
3	Harvest		Harvest	
4	Harvest			Harvest
5		Harvest		
6		Harvest	Harvest	
7		Harvest		Harvest
8			Harvest	
9			Harvest	Harvest
10				Harvest
11	No harvest			

Table 14.2. Harvest scheduling prescriptions with up to two possible harvests within a 40-year planning horizon and a minimum rotation of 20 years.

Prescription	Planning Period			
	1	2	3	4
1	Harvest			
2	Harvest		Harvest	
3	Harvest			Harvest
4		Harvest		
5		Harvest		Harvest
6			Harvest	
7				Harvest
8	No harvest			

minimum rotation of 20 years. These are the eight prescriptions we will consider in our four-period model.

Since up to two harvests are now allowed within the planning horizon, it is necessary to reconsider the notation for the basic decision variables of the model. For models with up to two harvests, the following decision variable definition will be used:

$X_{sap_1p_2}$ = the number of acres from site class s, initial age class a, assigned to be harvested first in period p_1 and again in period p_2.

Of course, p_2 should always be greater than p_1, and, if a minimum rotation of 20 years is required, p_2 will have to be greater than p_1 by at least 2. Similarly, for a minimum rotation of 30 years, p_2 will have to be at least 3 more than p_1. In the example presented below, p_2 is required to be greater than p_1 by at least 2. This is equivalent to requiring a minimum rotation on regenerated stands of at least 20 years. Obviously, if p_1 is zero p_2 must be zero also.

With the eight prescriptions shown in Table 14.2, the possible decision variables for the example problem are:

X_{sa10}, X_{sa13}, X_{sa14}, X_{sa20}, X_{sa24}, X_{sa30}, X_{sa40}, and X_{sa00};

where s = 1 or 2 and a = 1, 2, or 3.

For example, the variable X_{1324} represents the acres from site class 1, initial age class 3 that are assigned to be harvested in period 2 and again in period 4. These acres will be 40 years old the first time they are harvested and 20 years old at the second harvest.

Since there are six analysis areas and eight possible prescriptions, the problem will have 48 basic decision variables. There will also be a harvest accounting variable for each period:

H_1, H_2, H_3, and H_4.

This brings the total number of variables in the example problem to 52.

2. <u>The Objective Function for the Four-Period Profit-Maximization Model</u>

The general form of the objective function for the four-period, profit-maximization problem is:

$$Max\ Z = \sum_{s=1}^{2}\sum_{a=1}^{3}\sum_{p_1=1}^{4}\left[c^{p}_{sap_1 0}\cdot X_{sap_1 0} + \sum_{p_2=p_1+2}^{4} c^{p}_{sap_1p_2}\cdot X_{sap_1p_2}\right]$$

where $c^{p}_{sap_1p_2}$ = the discounted net revenue (profit) of assigning one acre from site class s, initial age class a to be harvested in periods p_1 and p_2

(where p_i = 0 implies no harvest); i.e., the discounted net revenue of assigning an acre to the variable $X_{sap_1p_2}$.

This equation appears especially complicated because of the additional summation inside the parentheses and because the period subscript for the second harvest is related to the period subscript for the first harvest. The first term inside the parentheses represents the variables where there is no second harvest. The summation inside the parentheses accounts for variables with two harvests. This summation only applies when $p_1 + 2$ # 4; otherwise, the counter will start at a value higher than the maximum counter value of four, which is the length of the planning horizon. Note that the terms with no harvests are not represented at all. They have been dropped since their coefficients are zero anyway, and they would complicate the formula unnecessarily. If the above equation seems overly complicated, it can be written more simply, but in a less compact and less general form as follows:

$$Max\ Z = \sum_{s=1}^{2}\sum_{a=1}^{3}[c_{sa10}^{p}X_{sa10} + c_{sa13}^{p}X_{sa13} + c_{sa14}^{p}X_{sa14} + c_{sa20}^{p}X_{sa20} + \\ + c_{sa10}^{p}X_{sa24} + c_{sa30}^{p}X_{sa30} + c_{sa40}^{p}X_{sa40}]$$

Here, the part in the parentheses explicitly lists the variables representing the seven prescriptions with harvests. This set of variables is repeated for each of the six analysis areas. Note that there will be 42 variables (with non-zero coefficients) in the objective function. As mentioned earlier, the discounted profit coefficient will be zero for prescriptions with no harvest (corresponding to the X_{sa00} variables), so they do not need to appear explicitly in the objective function. Similarly, the coefficients on the harvest variables (H_p) would also be zero. Of course, harvests do generate profits, but the value of wood that is harvested has already been accounted for by the coefficients associated with the analysis area prescription variables, and these profits would be double-counted if they were also associated with the harvest variables.

As in the two-period profit-maximization example, the objective function coefficients represent the discounted net revenue (profit) for each acre assigned to the corresponding variable. As before, when p_1 and p_2 are both zero, the coefficient will be zero. When there is just one harvest – i.e., when $p_1 > 0$ and $p_2 = 0$ – the calculation of the discounted net revenue will be just as before:

$$c_{sap_1 0}^{p} = \frac{(P - s_v)\cdot v_{sap_1 0}^{1} - E - s_f}{(1+r)^{10\cdot p_1 - 5}} \quad for\ p_1 > 0\ and\ p_2 = 0$$

where P = the wood price,

$v^{1}_{sap_1p_2}$ = the harvest volume per acre for the first harvest (in period p_1) from acres assigned to the variable $X_{sap_1p_2}$,

s_v = the variable (per cord) timber sale cost,

s_f = the fixed (per acre) timber sale cost,

E = the stand establishment (regeneration) cost, and
r = the real interest rate.

For variables with two harvests – i.e., when $p_1 > 0$ and $p_2 > 0$ – the calculation of the discounted net revenue will involve two terms, one for each harvest. Note, however, that each term is similar to the discounted net revenue expression when there is one harvest:

$$c^p_{sap_1p_2} = \frac{(P - s_v)\cdot v^1_{sap_1p_2} - E - s_f}{(1+r)^{10\cdot p_1 - 5}} + \frac{(P - s_v)\cdot v^2_{sap_1p_2} - E - s_f}{(1+r)^{10\cdot p_2 - 5}} \quad for\ p_1, p_2 > 0$$

where $v^1_{sap_1p_2}$ = the harvest volume per acre for the first harvest (in period p_1) from acres assigned to the variable $X_{sap_1p_2}$, and
$v^2_{sap_1p_2}$ = the harvest volume per acre for the second harvest (in period p_2) from acres assigned to the variable $X_{sap_1p_2}$.

As with the earlier formulations, the key to determining the harvest volume is to determine the age of the stand at harvest for each cut. For the first cut in a stand, the procedure for identifying the harvest age is the same as it was for the formulation with just one harvest: first, determine the age of the stand at the beginning of the planning horizon, then determine the year the harvest occurs. Remember that each activity is assumed to take place at the midpoint of the corresponding period. Add the initial age and the year in which the harvest occurs to get the age at harvest. To determine the age at harvest for the second cut in a stand, the logic is actually quite simple: take the difference between p_2 and p_1 and multiply this number by the period length (10 in this case).

As an example, the following formula demonstrates the calculation of the discounted net revenue coefficient for acres assigned to the variable X_{1324}.

$$c^p_{1324} = \frac{(25 - 0.2)\cdot 31 - 100 - 15}{(1.04)^{15}} + \frac{(25 - 0.2)\cdot 10 - 100 - 15}{(1.04)^{35}} = \$396.74$$

The first harvest occurs in period 2, when the stand is 40 years old. You can verify this by observing that the stand was initially 25 years old and it is scheduled to be cut 15 years later – at the midpoint of period 2. To determine the age at harvest for the second cut, note that the first cut is scheduled to occur in year 15 (the mid-point of period 2) and the stand is scheduled to be cut again in year 35 (the midpoint of period 4). Thus, the second cut in the stand will occur 20 years after the first cut, and the stand will be 20 years old at the time of the second harvest. The expected yield for a site I stand at age 40 is 31 cords per acre (see Table 14.2 in Chapter 12), and the expected yield for a site I stand at age 20 is 10 cords per acre.

The specific objective function for the four-period harvest scheduling problem is:

$$\begin{aligned}\text{Max } Z = & -53.75\,X_{1110} - 3.86\,X_{1113} + 42.80\,X_{1114} + 73.85\,X_{1120} + 107.55\,X_{1124} \\ & + 142.92\,X_{1130} + 165.68\,X_{1140} + 109.32\,X_{1210} + 159.21\,X_{1213} + 205.87\,X_{1214} \\ & + 211.56\,X_{1220} + 245.26\,X_{1224} + 245.25\,X_{1230} + 203.39\,X_{1240} + 313.15\,X_{1310} \\ & + 363.04\,X_{1313} + 409.71\,X_{1314} + 363.03\,X_{1320} + 396.74\,X_{1324} + 301.07\,X_{1330}\end{aligned}$$

$$+ 234.81\,X_{1340} + 7.40\,X_{2110} + 94.50\,X_{2113} + 147.94\,X_{2114} + 128.93\,X_{2120}$$
$$+ 187.78\,X_{2124} + 208.04\,X_{2130} + 209.68\,X_{2140} + 190.85\,X_{2210} + 277.95\,X_{2213}$$
$$+ 331.40\,X_{2214} + 307.95\,X_{2220} + 366.79\,X_{2224} + 310.37\,X_{2230} + 266.24\,X_{2240}$$
$$+ 455.84\,X_{2310} + 542.94\,X_{2313} + 596.39\,X_{2314} + 459.43\,X_{2320} + 518.27\,X_{2324}$$
$$+ 394.10\,X_{2330} + 310.23\,X_{2340}$$

3. The Constraints of the Four-Period Harvest Scheduling Model

As with the two-period models, there are four basic types of constraints in the four-period harvest scheduling model: 1) area constraints, 2) harvest constraints, 3) ending age constraints, and 4) the non-negativity constraints. The area constraints are straightforward; the sum of the acres from a given analysis area assigned to each prescription must be less than or equal to the number of acres in the analysis area. As in Chapter 13, harvest fluctuation constraints will be used in this chapter's example since it is a profit maximizing model. With more than two periods, the number of ways that the harvest fluctuation constraints can be applied increases. Also, specifying the harvest accounting constraints is somewhat more complicated. The constraint on the average age of the forest at the end of the planning horizon is a fairly straightforward extension of the average ending age constraints that were used in the two-period models. Two ending average age constraints will be formulated for this model, however. The original constraint will be separated into two: one constraint for each site class.

Area Constraints

The area constraints state that the sum of the acres assigned to each of the prescriptions from each analysis area must be less than or equal to the available acres in that analysis area. To construct these constraints, simply list the variables representing each of the possible prescriptions for each analysis area. For the four-period model, the general form of the area constraints is:

$$X_{sa00} + \sum_{p_1=1}^{4}\left[X_{sap_1 0} + \sum_{p_2=p_1+2}^{4} X_{sap_1 p_2} \right] \le A_{sa} \quad \text{for } s = 1,2 \text{ and } a = 1,2,3$$

A less general, but easier to understand, expression for these constraints is:

$$X_{sa00} + X_{sa10} + X_{sa13} + X_{sa14} + X_{sa20} + X_{sa24} + X_{sa30} + X_{sa40} \;\#\; A_{sa}$$

for s = 1, 2 and a = 1, 2, 3

The specific area constraints for the four-period, profit-maximization example problem are:

$$X_{1100} + X_{1110} + X_{1113} + X_{1114} + X_{1120} + X_{1124} + X_{1130} + X_{1140} \;\#\; 3000$$
$$X_{1200} + X_{1210} + X_{1213} + X_{1214} + X_{1220} + X_{1224} + X_{1230} + X_{1240} \;\#\; 6000$$
$$X_{1300} + X_{1310} + X_{1313} + X_{1314} + X_{1320} + X_{1324} + X_{1330} + X_{1340} \;\#\; 9000$$
$$X_{2100} + X_{2110} + X_{2113} + X_{2114} + X_{2120} + X_{2124} + X_{2130} + X_{2140} \;\#\; 8000$$

$X_{2200} + X_{2210} + X_{2213} + X_{2214} + X_{2220} + X_{2224} + X_{2230} + X_{2240}$ # 4000
$X_{2300} + X_{2310} + X_{2313} + X_{2314} + X_{2320} + X_{2324} + X_{2330} + X_{2340}$ # 7000

Harvest Accounting Constraints

The harvest accounting constraint for each period must include all of the variables involving a harvest in that period. The complicating factor with longer planning horizons is that acres assigned to some variables will provide volume in more than one period. Thus, some variables will have non-zero coefficients in more than one harvest accounting constraint. As before, the coefficients in these constraints give the volume of wood produced in the period for each acre assigned to the corresponding variable.

The variables involving a harvest in period 1 are:

X_{sa10}, X_{sa13}, and X_{sa14} for $s = 1$ or 2 and $a = 1, 2$, or 3.

For all of the variables, the harvest in period 1 is the first harvest from the stand. The harvest accounting constraint for period 1 will therefore have the following general form:

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa10}^{1}X_{sa10} + v_{sa13}^{1}X_{sa13} + v_{sa14}^{1}X_{sa14}\right] - H_1 = 0$$

The specific harvest accounting constraint for period 1 for the example problem is:

$$\begin{aligned} &2\,X_{1110} + 2\,X_{1113} + 2\,X_{1114} + 10\,X_{1210} + 10\,X_{1213} + 10\,X_{1214} \\ &\quad + 20\,X_{1310} + 20\,X_{1313} + 20\,X_{1314} + 5\,X_{2110} + 5\,X_{2113} + 5\,X_{2114} \\ &\quad + 14\,X_{2210} + 14\,X_{2213} + 14\,X_{2214} + 27\,X_{2310} + 27\,X_{2313} + 27\,X_{2314} - H_1 = 0 \end{aligned}$$

The variables with a harvest in period 2 are:

X_{sa20}, and X_{sa24} for $s = 1$ or 2 and $a = 1, 2$, or 3.

Again, for each of these variables, the harvest in period 2 will be the first harvest in the stand. The general form of the harvest accounting constraint for period 2 is:

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa20}^{1}X_{sa20} + v_{sa24}^{1}X_{sa24}\right] - H_2 = 0$$

The specific harvest accounting constraint for period 2 for the example problem is:

$$\begin{aligned} &10\,X_{1120} + 10\,X_{1124} + 20\,X_{1220} + 20\,X_{1224} + 31\,X_{1320} + 31\,X_{1324} \\ &\quad + 14\,X_{2120} + 14\,X_{2124} + 27\,X_{2220} + 27\,X_{2224} + 38\,X_{2320} + 38\,X_{2324} - H_2 = 0 \end{aligned}$$

The variables with a harvest in period 3 are:

X_{sa30}, and X_{sa13} for $s = 1$ or 2 and $a = 1, 2$, or 3.

Note that for the variables of the form X_{sa30}, the harvest in the third period is the first harvest in the stand. For the variables of the form X_{sa13}, the harvest in the third period is the second harvest in the stand. The general harvest accounting constraint for period 3 is:

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa30}^{1}X_{sa30}+v_{sa13}^{2}X_{sa13}\right]-H_3=0$$

The specific harvest accounting constraint for period 3 for the example problem is:

$$10\,X_{1113}+20\,X_{1130}+10\,X_{1213}+31\,X_{1230}+10\,X_{1313}+37\,X_{1330} + 14\,X_{2113}+27\,X_{2130}+14\,X_{2213}+38\,X_{2230}+14\,X_{2313}+47\,X_{2330}-H_3=0$$

The variables involving a harvest in period 4 are:

X_{sa40}, X_{sa14}, and X_{sa24} for $s = 1$ or 2 and $a = 1, 2,$ or 3.

The general form of the harvest accounting constraint for period 4 is:

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa40}^{1}X_{sa40}+v_{sa14}^{2}X_{sa14}+v_{sa24}^{2}X_{sa24}\right]-H_4=0$$

The specific harvest accounting constraints for period 4 for the example problem is:

$$20\,X_{1114}+10\,X_{1124}+31\,X_{1140}+20\,X_{1214}+10\,X_{1224}+37\,X_{1240}+20\,X_{1314}+10\,X_{1324} + 42\,X_{1340}+27\,X_{2114}+14\,X_{2124}+38\,X_{2140}+27\,X_{2214}+14\,X_{2224}+47\,X_{2240} + 27\,X_{2314}+14\,X_{2324}+54\,X_{2340}-H_4=0$$

Harvest Fluctuation Constraints

With only two periods in the planning horizon the harvest fluctuation constraints are fairly straightforward: the only question is how much to allow the harvest to go up or down between the two periods. With more periods, there are more possibilities. For example, should the harvest in period 3 be tied to the harvest in period 1 directly, or should we only be concerned with fluctuations between adjacent periods? If we only link harvests in adjacent periods, the harvest can go down one period after another until the overall change in the harvest level over the planning horizon becomes quite substantial.

It is up to you, the forest manager, to decide whether harvest fluctuation restrictions should apply only to adjacent periods or to all combinations of periods in your model. There is no right or wrong way to do it. For example, you could choose to link the harvest levels for adjacent periods only, allowing a maximum change of 10% from one period to the next. In this case, the period 2 harvest would not be allowed to vary from the period 1 harvest by more than 10%, and the period 3 harvest would not be allowed to vary from the period 2 harvest by more than 10%. With harvests tied together only for adjacent periods, harvests could go down (or up) by 10% in two consecutive periods. This would result in a total fluctuation over the two periods of approximately 20%. For example, if the period 1 harvest was 10,000 cords, then the period 2 harvest would have to be between 9,000 and 11,000 cords. If the harvest in period 2 was 9,000 cords, the harvest in period 3 could be as low as 8,100 cords.

If you choose to link the harvests in adjacent periods only, the specific harvest fluctuation constraints for the example problem would be:

$$0.9\, H_1 - H_2 \# 0$$
$$-\, 1.1\, H_1 + H_2 \# 0$$
$$0.9\, H_2 - H_3 \# 0$$
$$-\, 1.1\, H_2 + H_3 \# 0$$
$$0.9\, H_3 - H_4 \# 0$$
$$-\, 1.1\, H_3 + H_4 \# 0$$

Note that six constraints are required – two for each pair of adjacent periods. The first constraint keeps the harvest in period 2 from declining by more than 10% from the harvest in period 1. The second constraint prevents the harvest from increasing by more than 10% between the first two periods. The next pair of constraints prevents the harvest level from changing by more than 10% between periods 2 and 3. Similarly, the fifth and sixth constraints limit the amount of change allowed between periods 3 and 4.

If all periods were tied together through harvest fluctuation constraints, the harvest volume in each period would be required to be within 10% of the harvest volume in all other periods. Thus, if the harvest in period 1 was 10,000 cords, then all future harvests would have to be between 9,000 and 11,000 cords. If the harvest in period 2 was 9,000 cords, the harvests in all of the remaining periods would have to be between 9,000 and 9,900 cords. To implement these restrictions, we would need additional pairs of constraints limiting the difference between the harvest levels in periods 1 and 3, 1 and 4, and 2 and 4. This would require the following six <u>additional</u> constraints:

$$0.9\, H_1 - H_3 \# 0$$
$$-\, 1.1\, H_1 + H_3 \# 0$$
$$0.9\, H_1 - H_4 \# 0$$
$$-\, 1.1\, H_1 + H_4 \# 0$$
$$0.9\, H_2 - H_4 \# 0$$
$$-\, 1.1\, H_2 + H_4 \# 0$$

Whether or not to tie the harvest in each period to the harvest in all other periods is only one possible decision. One could also choose to require harvest fluctuations to decline over time, or harvests could be allowed to increase but not to decrease. This latter type of constraint is called non-declining flow. A problem with non-declining flow is that the harvest may decline after the planning horizon if the ending conditions are not chosen carefully. In this case, the objective of non-declining flow will only have been met during the planning horizon, and the declines after the end of the planning horizon may be unintended. The following set of constraints prevent the harvest level from decreasing during the planning horizon and allow increases in the harvest level of up to 10%.

H_1 - H_2 # 0
- 1.1 H_1 + H_2 # 0
H_2 - H_3 # 0
- 1.1 H_2 + H_3 # 0
H_3 - H_4 # 0
- 1.1 H_3 + H_4 # 0

For the example problem discussed in this chapter, we will constrain harvest level changes only between adjacent periods and require the change in the harvest level from one period to the next to be less than or equal to 10 percent.

Average Ending Age Constraints by Site Class

The average ending age constraint is basically the same as in the two-period model. All of the variables are included on the left-hand side, and the coefficient on each variable is the age at the end of the time horizon of acres assigned to that variable. If the acres are to be harvested some time – maybe more than once – during the time horizon, their age will depend only on the elapsed time since the last harvest. If the acres are not to be harvested during the time horizon, then they will be forty years older than they were at the beginning of the time horizon.

The general form of the ending age constraint for a four-period model is:

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[Age_{sa00}^{40}X_{sa00} + \sum_{p_1=1}^{4}\left(Age_{sap_10}^{40}X_{sap_10} + \sum_{p_2=p_1+2}^{4} Age_{sap_1p_2}^{40}X_{sap_1p_2}\right)\right] \geq \overline{Age}^{40} \times TotalArea$$

We are not going to use this constraint in exactly this form, however. Often, the harvest scheduling model will allow a poorer site class to grow old so that the better sites can be over-harvested while the average ending age constraint is still met. In order to prevent this type of solution, it is sometimes necessary to use separate ending age constraints for each site class. This requires as many ending age constraints as there are site classes – two in this example. These constraints would be exactly the same as the above formula, except that the first summation sign would be dropped. These constraints can also be written equivalently as follows:

$$\sum_{a=1}^{3}\left[Age_{sa00}^{40}X_{sa00} + Age_{sa10}^{40}X_{sa10} + Age_{sa13}^{40}X_{sa13} + Age_{sa14}^{40}X_{sa14} + Age_{sa20}^{40}X_{sa20} + Age_{sa24}^{40}X_{sa24} + Age_{sa30}^{40}X_{sa30} + Age_{sa40}^{40}X_{sa40}\right] \geq \overline{Age}_s^{40} \times Area_s$$

There will be one of these constraints for each site class. Note that splitting the ending age constraint into two allows the target average ending age to be different for different site classes. This is desirable in this example, since the optimal rotation is different for the different site classes.

The specific average ending age constraints for the example model are:

$$
\begin{aligned}
&35\,X_{1110} + 15\,X_{1113} + 5\,X_{1114} + 25\,X_{1120} + 5\,X_{1124} + 15\,X_{1130} + 5\,X_{1140} + 35\,X_{1210}\\
&+ 15\,X_{1213} + 5\,X_{1214} + 25\,X_{1220} + 5\,X_{1224} + 15\,X_{1230} + 5\,X_{1240} + 35\,X_{1310} + 15\,X_{1313}\\
&+ 5\,X_{1314} + 25\,X_{1320} + 5\,X_{1324} + 15\,X_{1330} + 5\,X_{1340} + 45\,X_{1100} + 55\,X_{1200} + 65\,X_{1300}\\
&\qquad \$\ 369{,}000\\
&35\,X_{2110} + 15\,X_{2113} + 5\,X_{2114} + 25\,X_{2120} + 5\,X_{2124} + 15\,X_{2130} + 5\,X_{2140} + 35\,X_{2210}\\
&+ 15\,X_{2213} + 5\,X_{2214} + 25\,X_{2220} + 5\,X_{2224} + 15\,X_{2230} + 5\,X_{2240} + 35\,X_{2310} + 15\,X_{2313}\\
&+ 5\,X_{2314} + 25\,X_{2320} + 5\,X_{2324} + 15\,X_{2330} + 5\,X_{2340} + 45\,X_{2100} + 55\,X_{2200} + 65\,X_{2300}\\
&\qquad \$\ 294{,}500
\end{aligned}
$$

The right-hand sides of these constraints are based on a target average ending age of 20.5 years for site class I and 15.5 years for site class II. For example, the value 369,000 was obtained by multiplying the target ending age for site class I, 20.5, times the area in site class I, 18,000 acres. Note that, with a forty-year planning horizon, there should be plenty of time for the forest to reach the average age of an optimally regulated forest.

Non-negativity Constraints

The non-negativity constraints for this problem are:

$X_{sap_1p_2}$ \$0 for s = 1, 2; a = 1, 2, 3; and p_1p_2 = 00, 10, 13, 14, 20, 24, 30, and 40; and

H_p \$ 0 for p = 1, 2, 3, 4.

5. The Complete Four-Period Profit-Maximization Harvest Scheduling Formulation

This section summarizes the problem formulation for the four-period profit-maximization harvest scheduling model. First, the general formulation of the model is summarized; then, the specific formulation for this example is summarized.

General Formulation

$$
Max\ Z = \sum_{s=1}^{2}\sum_{a=1}^{3}\sum_{p_1=1}^{4}\left[c^{p}_{sap_1 0}\cdot X_{sap_1 0} + \sum_{p_2=p_1+2}^{4} c^{p}_{sap_1p_2}\cdot X_{sap_1p_2}\right]
$$

Subject to

$$X_{sa00} + \sum_{p=1}^{4}\left[X_{sap_1 0} + \sum_{p_2=p_1+2}^{4} X_{sap_1 p_2}\right] \geq A_{sa} \quad \text{for } s=1,2 \text{ and } a=1,2,3$$

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa10}^{1}X_{sa10} + v_{sa13}^{1}X_{sa13} + v_{sa14}^{1}X_{sa14}\right] - H_1 = 0$$

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa20}^{1}X_{sa20} + v_{sa24}^{1}X_{sa24}\right] - H_2 = 0$$

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa30}^{1}X_{sa30} + v_{sa13}^{2}X_{sa13}\right] - H_3 = 0$$

$$\sum_{s=1}^{2}\sum_{a=1}^{3}\left[v_{sa40}^{1}X_{sa40} + v_{sa14}^{2}X_{sa14} + v_{sa24}^{2}X_{sa24}\right] - H_4 = 0$$

0.9 H_1 - H_2 # 0
- 1.1 H_1 + H_2 # 0
0.9 H_2 - H_3 # 0
- 1.1 H_2 + H_3 # 0
0.9 H_3 - H_4 # 0
- 1.1 H_3 + H_4 # 0

$$\sum_{a=1}^{3}\left[Age_{sa00}^{40}X_{sa00} + \sum_{p_1=1}^{4}\left(Age_{sap_1 0}^{40}X_{sap_1 0} + \sum_{p_2=p_1+2}^{4} Age_{sap_1 p_2}^{40}X_{sap_1 p_2}\right)\right] \geq \overline{Age}^{40} \times Area_{Site\,I} \quad for\ s=1,2$$

$X_{sap_1p_2}$ \$0 for s = 1, 2; a = 1, 2, 3; and p_1p_2 = 00, 10, 13, 14, 20, 24, 30, and 40;

and

H_p \$ 0 for p = 1, 2, 3, 4.

Specific Formulation

Max Z = -53.75 X_{1110} - 3.86 X_{1113} + 42.80 X_{1114} + 73.85 X_{1120} + 107.55 X_{1124}
+ 142.92 X_{1130} + 165.68 X_{1140} + 109.32 X_{1210} + 159.21 X_{1213} + 205.87 X_{1214}
+ 211.56 X_{1220} + 245.26 X_{1224} + 245.25 X_{1230} + 203.39 X_{1240} + 313.15 X_{1310}
+ 363.04 X_{1313} + 409.71 X_{1314} + 363.03 X_{1320} + 396.74 X_{1324} + 301.07 X_{1330}
+ 234.81 X_{1340} + 7.40 X_{2110} + 94.50 X_{2113} + 147.94 X_{2114} + 128.93 X_{2120}
+ 187.78 X_{2124} + 208.04 X_{2130} + 209.68 X_{2140} + 190.85 X_{2210} + 277.95 X_{2213}
+ 331.40 X_{2214} + 307.95 X_{2220} + 366.79 X_{2224} + 310.37 X_{2230} + 266.24 X_{2240}
+ 455.84 X_{2310} + 542.94 X_{2313} + 596.39 X_{2314} + 459.43 X_{2320} + 518.27 X_{2324}
+ 394.10 X_{2330} + 310.23 X_{2340}

Subject to

$$X_{1110} + X_{1113} + X_{1114} + X_{1120} + X_{1124} + X_{1130} + X_{1140} + X_{1100} \# 3000$$
$$X_{1210} + X_{1213} + X_{1214} + X_{1220} + X_{1224} + X_{1230} + X_{1240} + X_{1200} \# 6000$$
$$X_{1310} + X_{1313} + X_{1314} + X_{1320} + X_{1324} + X_{1330} + X_{1340} + X_{1300} \# 9000$$
$$X_{2110} + X_{2113} + X_{2114} + X_{2120} + X_{2124} + X_{2130} + X_{2140} + X_{2100} \# 8000$$
$$X_{2210} + X_{2213} + X_{2214} + X_{2220} + X_{2224} + X_{2230} + X_{2240} + X_{2200} \# 4000$$
$$X_{2310} + X_{2313} + X_{2314} + X_{2320} + X_{2324} + X_{2330} + X_{2340} + X_{2300} \# 7000$$
$$2\,X_{1110} + 2\,X_{1113} + 2\,X_{1114} + 10\,X_{1210} + 10\,X_{1213} + 10\,X_{1214} + 20\,X_{1310} + 20\,X_{1313}$$
$$+ 20\,X_{1314} + 5\,X_{2110} + 5\,X_{2113} + 5\,X_{2114} + 14\,X_{2210} + 14\,X_{2213} + 14\,X_{2214} + 27\,X_{2310}$$
$$+ 27\,X_{2313} + 27\,X_{2314} - H_1 = 0$$
$$10\,X_{1120} + 10\,X_{1124} + 20\,X_{1220} + 20\,X_{1224} + 31\,X_{1320} + 31\,X_{1324}$$
$$+ 14\,X_{2120} + 14\,X_{2124} + 27\,X_{2220} + 27\,X_{2224} + 38\,X_{2320} + 38\,X_{2324} - H_2 = 0$$
$$10\,X_{1113} + 20\,X_{1130} + 10\,X_{1213} + 31\,X_{1230} + 10\,X_{1313} + 37\,X_{1330}$$
$$+ 14\,X_{2113} + 27\,X_{2130} + 14\,X_{2213} + 38\,X_{2230} + 14\,X_{2313} + 47\,X_{2330} - H_3 = 0$$
$$20\,X_{1114} + 10\,X_{1124} + 31\,X_{1140} + 20\,X_{1214} + 10\,X_{1224} + 37\,X_{1240} + 20\,X_{1314} + 10\,X_{1324}$$
$$+ 42\,X_{1340} + 27\,X_{2114} + 14\,X_{2124} + 38\,X_{2140} + 27\,X_{2214} + 14\,X_{2224} + 47\,X_{2240} + 27\,X_{2314}$$
$$+ 14\,X_{2324} + 54\,X_{2340} - H_4 = 0$$
$$0.9\,H_1 - H_2 \# 0$$
$$-1.1\,H_1 + H_2 \# 0$$
$$0.9\,H_2 - H_3 \# 0$$
$$-1.1\,H_2 + H_3 \# 0$$
$$0.9\,H_3 - H_4 \# 0$$
$$-1.1\,H_3 + H_4 \# 0$$
$$35\,X_{1110} + 15\,X_{1113} + 5\,X_{1114} + 25\,X_{1120} + 5\,X_{1124} + 15\,X_{1130} + 5\,X_{1140} + 35\,X_{1210}$$
$$+ 15\,X_{1213} + 5\,X_{1214} + 25\,X_{1220} + 5\,X_{1224} + 15\,X_{1230} + 5\,X_{1240} + 35\,X_{1310} + 15\,X_{1313}$$
$$+ 5\,X_{1314} + 25\,X_{1320} + 5\,X_{1324} + 15\,X_{1330} + 5\,X_{1340} + 45\,X_{1100} + 55\,X_{1200} + 65\,X_{1300}$$
$$\$\ 369{,}000$$
$$35\,X_{2110} + 15\,X_{2113} + 5\,X_{2114} + 25\,X_{2120} + 5\,X_{2124} + 15\,X_{2130} + 5\,X_{2140} + 35\,X_{2210}$$
$$+ 15\,X_{2213} + 5\,X_{2214} + 25\,X_{2220} + 5\,X_{2224} + 15\,X_{2230} + 5\,X_{2240} + 35\,X_{2310} + 15\,X_{2313}$$
$$+ 5\,X_{2314} + 25\,X_{2320} + 5\,X_{2324} + 15\,X_{2330} + 5\,X_{2340} + 45\,X_{2100} + 55\,X_{2200} + 65\,X_{2300}$$
$$\$\ 294{,}500$$

$X_{sap_1p_2}$ \$0 for s = 1, 2; a = 1, 2, 3; and p_1p_2 = 00, 10, 13, 14, 20, 24, 30, and 40;

H_p \$ 0 for p = 1, 2, 3, 4.

6. Interpreting the Solution to the Four-Period Model

The LINDO output for the four-period profit-maximization model is presented in Figure 14.1. Tables 14.3 through 14.9 organize and present the results.

Table 14.3 shows the acres harvested by period from each analysis area. Note that the total of the acres harvested from an analysis area over all four periods need not equal the number

```
 LP OPTIMUM FOUND AT STEP     33
        OBJECTIVE FUNCTION VALUE
        1)      .1191013E+08

  VARIABLE        VALUE          REDUCED COST
     X1110          .000000        124.680100
     X1113          .000000        144.128500
     X1114          .000000        124.680100
     X1120          .000000         34.489690
     X1124          .000000         65.899490
     X1130      1032.951000           .000000
     X1140      1967.049000           .000000
     X1210          .000000         59.081680
     X1213          .000000         78.530110
     X1214          .000000         59.081680
     X1220      4383.481000           .000000
     X1224          .000000         31.409830
     X1230      1616.519000           .000000
     X1240          .000000         61.095940
     X1310      2850.115000           .000000
     X1313          .000000         19.448340
     X1314      2183.274000           .000000
     X1320      3966.611000           .000000
     X1324          .000000         31.409710
     X1330          .000000         93.475160
     X1340          .000000        176.654000
     X2110          .000000         95.882030
     X2113          .000000        109.286700
     X2114          .000000         95.881990
     X2120          .000000         29.178310
     X2124          .000000         64.927150
     X2130      8000.000000           .000000
     X2140          .000000         33.356710
     X2210          .000000         55.267910
     X2213          .000000         68.672610
     X2214          .000000         55.267940
     X2220      4000.000000           .000000
     X2224          .000000         35.748860
     X2230          .000000         45.702750
     X2240          .000000        120.629400
     X2310      1316.667000           .000000
     X2313          .000000         13.404730
     X2314      5683.333000           .000000
     X2320          .000000         65.968570
     X2324          .000000        101.717500
     X2330          .000000        177.833500
     X2340          .000000        289.138900
     X1100          .000000         37.918370
     X1200          .000000        104.389500
     X1300          .000000        218.814100
     X2100          .000000         56.088310
     X2200          .000000        153.075300
     X2300          .000000        318.281600
        H1    289667.800000           .000000
        H2    318634.600000           .000000
        H3    286771.100000           .000000
        H4    258094.000000           .000000
```

Figure 14.1. LINDO solution to the four-period profit-maximization harvest scheduling problem.

Table 14.3. Acres harvested by analysis area and period.

Site Class	Initial Age Class	Harvest Period 1	2	3	4	No cut
1	0-10	0.0	0.0	1,033.0	1,967.0	0.0
	11-20	0.0	4,383.5	1,616.5	0.0	0.0
	21-30	5,033.4	3,966.6	0.0	2,183.3	0.0
	Total	5,033.4	8,350.1	2,649.5	4,150.3	0.0
2	0-10	0.0	0.0	8,000.0	0.0	0.0
	11-20	0.0	4,000.0	0.0	0.0	0.0
	21-30	7,000.0	0.0	0.0	5,683.3	0.0
	Total	7,000.0	4,000.0	8,000.0	5,683.3	0.0
	TOTAL	12,033.4	12,350.1	10,649.5	9,833.7	0.0

of acres initially in the analysis area. For example, summing across the row for analysis area 1,3 (i.e., site class 1, initial age class 3, ages 21 to 30 years) gives a total area cut of 11,183.3 acres, yet there were only 9,000 acres in the analysis area to begin with. This is because 2,183.3 acres from this analysis area were cut twice, so they appear two times in the table. Note also that there are no longer any uncut acres. Over a 40-year planning horizon, the model schedules all acres to be cut at least once.

Table 14.4. Harvest schedule - acres harvested by period and age at harvest.

Planning Period	Age at harvest	Site 1	2	Total
1	30	5,033.4	7,000.0	12,033.4
	Total	5,033.4	7,000.0	12,033.4
2	30	4,383.5	4,000.0	8,383.5
	40	3,966.6	0.0	3,966.6
	Total	8,350.1	4,000.0	12,350.1
3	30	1,033.0	8,000.0	9,033.0
	40	1,616.5	0.0	1,616.5
	Total	2,649.5	8,000.0	10,649.5
4	30	2,183.3	5,683.3	7,866.6
	40	1,967.0	0.0	1,967.0
	Total	4,150.3	5,683.3	9,833.7

The information presented in Table 14.3 is not particularly useful to the forester charged with implementing the plan. It is very hard to tell, for example, when one gets to period 4, where the 2,183.3 acres to be cut from analysis area 1,3 should be found. By the fourth period, these acres will have been cut once, and they will be relatively young, at 30 years, and not seem like they were 30 years old 35 years before. The information in the harvest schedule table (Table 14.4) will be much more useful. That table shows the acres and the age classes where harvests should occur in terms of their ages in that period. Thus, in Table 14.4, the 2,183.3 acres to be cut from the original analysis area 1,3 are identified as being 30 years old in period 4.

Table 14.5 shows the summary of acres and volume harvested and costs and revenue by period. Recall the general rule that the planning horizon is long enough if extending the planning horizon by one period does not significantly change the planned activities for period 1. Although the planning horizon has been extended by more than one period, and the formulation of the ending age constraints have been modified somewhat, it should still be interesting to compare the general results from the four-period model developed in this chapter with the results from the two-period model developed in Chapter 13. In comparing the results in this chapter with the results in the previous chapter, you should note that the total area scheduled for harvest in period one has declined from 12,895 acres to 12,033 acres, a 6.7 percent decrease. All of this decrease is scheduled to occur in the acres cut from site class I; in both models, all of the acres in analysis area 2,3 (site class 2, initial age class 3, aged 21 to 30) are scheduled to be harvested in period one. Similarly, the volume scheduled to be harvested in period 1 has been reduced from 30,691 cords per year to 28,967 cords per year, a 6.4 percent decrease. There is no purely objective way to say whether these changes are significant; however, most would consider this a fairly substantial change in the plan. If so, in this case, the four-period model should be considered to be a worthwhile improvement over the two-period model of Chapter 13.

Table 14.5. Summary table: acres and volume harvested and revenues and costs by period.

	Period			
Item	1	2	3	4
Acres	12,033	12,350	10,649	9,834
Volume	289,668	318,635	286,771	258,094
Planting	1,203,339	1,235,009	1,064,947	983,366
Timber Sales	238,434	248,978	217,096	199,124
Revenue	7,241,695	7,965,865	7,169,277	6,452,350
Net Revenue	5,799,922	6,481,877	5,887,234	5,269,861
Disc. NR	4,767,113	3,599,156	2,208,400	1,335,464

Another interesting observation that can be made regarding the results in Table 14.5 is that the model has apparently taken full advantage of the flexibility in the harvest volume from period to period that is allowed by the harvest fluctuation constraints. Between periods 1 and 2, the harvest volume increases by 10 percent. Then, the harvest volume decreases by 10 percent between periods 2 and 3 and again between periods 3 and 4. Recall from Chapter 12, that the LTSY of this forest is 31,050 cords per year. Rather than approaching this harvest level in the long run, the harvest in the fourth period is almost 17 percent below the LTSY. This outcome is probably not desirable, and the formulation of the harvest target constraints for this model should probably be reconsidered. Thus, while we only have time to consider this initial formulation of the model in this book, you should recognize that in practice it will frequently be necessary to reformulate the model several times before a completely satisfactory plan has been developed.

Table 14.6. Age-class distribution at the end of period 1.

Age Classes	Acres by site class		
	Site I	Site II	Total
0 to 10	5,033.4	7,000.0	12,033.4
11 to 20	3,000.0	8,000.0	11,000.0
21 to 30	6,000.0	4,000.0	10,000.0
31 to 40	3,966.6	0.0	3,966.6
Total	18,000.0	19,000.0	37,000.0

Table 14.7. Age-class distribution at the end of period 2.

Age Classes	Acres by site class		
	Site I	Site II	Total
0 to 10	8,350.1	4,000.0	12,350.1
11 to 20	5,033.4	7,000.0	12,033.4
21 to 30	3,000.0	8,000.0	11,000.0
31 to 40	1,616.5	0.0	1,616.5
Total	18,000.0	19,000.0	37,000.0

Table 14.8. Age-class distribution at the end of period 3.

Age Classes	Acres by site class		
	Site I	Site II	Total
0 to 10	2,649.5	8,000.0	10,649.5
11 to 20	8,350.1	4,000.0	12,350.1
21 to 30	5,033.4	7,000.0	12,033.4
31 to 40	1,967.0	0.0	1,967.0
Total	18,000.0	19,000.0	37,000.0

Table 14.9. Age-class distribution at the end of period 4.

Age Classes	Acres by site class		
	Site I	Site II	Total
0 to 10	4,150.3	5,683.3	9,833.7
11 to 20	2,649.5	8,000.0	10,649.5
21 to 30	8,350.1	4,000.0	12,350.1
31 to 40	2,850.1	1,316.7	4,166.8
Total	18,000.0	19,000.0	37,000.0

Tables 14.6 through 14.9 show the projected evolution of the age-class distribution of the forest over time. Note that there is really no trend towards a more regulated age-class distribution. Thus, if this is an objective, you would need to consider how the formulation might be modified to achieve this. On the other hand, if the primary reason for wanting to balance the age-class distribution is to achieve a more even-flow of timber harvests, then this would probably be accomplished best by modifying the harvest fluctuation constraints.

7. Study Questions

1. What are the two basic guidelines for selecting the length of the planning horizon for a harvest scheduling linear program?

2. What are the advantages and disadvantages of longer versus shorter planning horizons?

3. List the possible harvesting combinations (prescriptions) with a fifty-year planning horizon (consisting of five 10-year planning periods) if up to three harvests are allowed within the planning horizon and if at least two periods (twenty years) must pass between harvests?

4. Use the definition presented in this chapter to explain how to interpret the variable X_{2113}.

5. The objective function coefficient on the variable X_{2124} is 187.78. What does this number represent? Show how this coefficient was calculated.

6. How old will acres assigned to the variable X_{1214} be the first time they are harvested? What will be the per-acre yield from this harvest? How old will the same acres be the second time they are harvested? What will the yield be from this second harvest?

7. What is the purpose of the harvest fluctuation constraints? What is the difference between tying together the harvest level only in adjacent periods and linking the harvest levels in all periods? With a five-period model, how many constraints would be required to link the harvest levels of adjacent periods? How many constraints would be required to link the harvest levels in all periods with a five-period model?

8. Why is it sometimes advisable to have separate ending constraints for each site class?