The Simplex method at a glance

1 Mar 2019

A pulp mill makes mechanical and chemical pulp and during the production process it pollutes the river in which it spills its spent waters. The owners would like to minimize pollution, keeping at least 300 people employed at the mill and generating at least 40000€ of revenue per day.

Keeping the river clean - Data

- The maximum capacity of the mill is 300 tons per day to make mechanical pulp and 200 tons per day to make chemical pulp (the mechanical pulp line cannot be used to make chemical pulp, and vice-versa)
- Both mechanical and chemical pulp require the labor of 1 worker for about 1 day, or 1 workday (wd), per ton produced
- Pollution is measured by the biological oxygen demand (BOD). 1 ton of mechanical pulp produces 1 unit of BOD, 1 ton of chemical pulp produces 1.5 units
- The chemical pulp sells at 200€, the mechanical pulp at 100€ per ton.

Keeping the river clean - Decision variables and formulation

x_1 - Amount of mechanical pulp produced (in tons/day, or t/d)

x_2 - Amount of chemical pulp produced (t/d).

$$\min Z = x_1 + 1.5x_2 \quad \text{units of BOD per day} \tag{1}$$

$$\textit{subject to}$$

$$x_1 + x_2 \geq 300 \quad \text{workers employed} \tag{2}$$

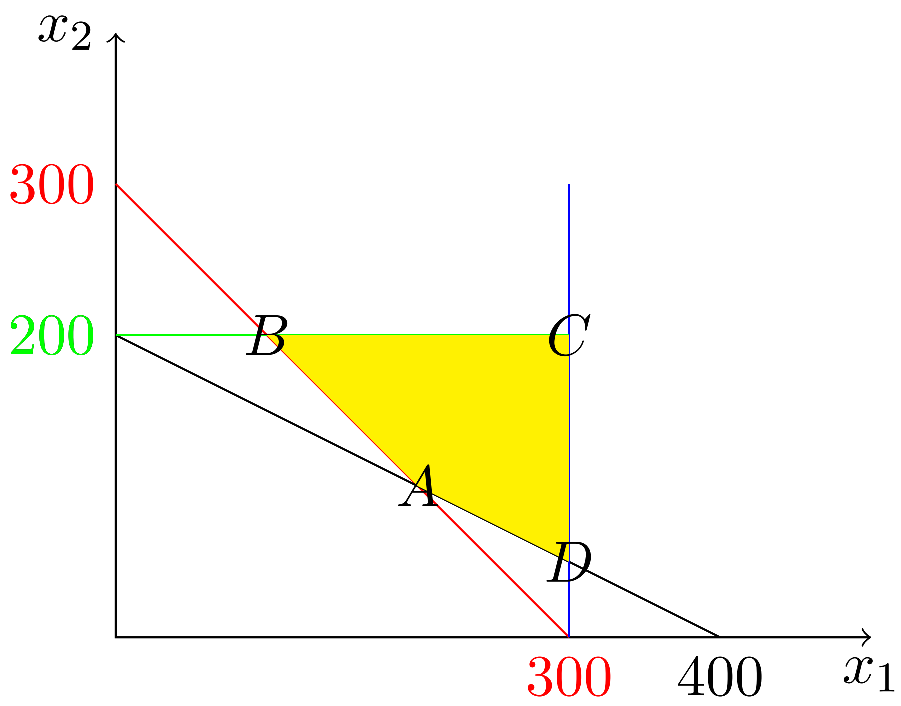
$$100x_1 + 200x_2 \geq 40000 \quad \text{revenue, euros/day} \tag{3}$$

$$x_1 \leq 300 \quad \text{mechanical pulping capacity, t/day} \tag{4}$$

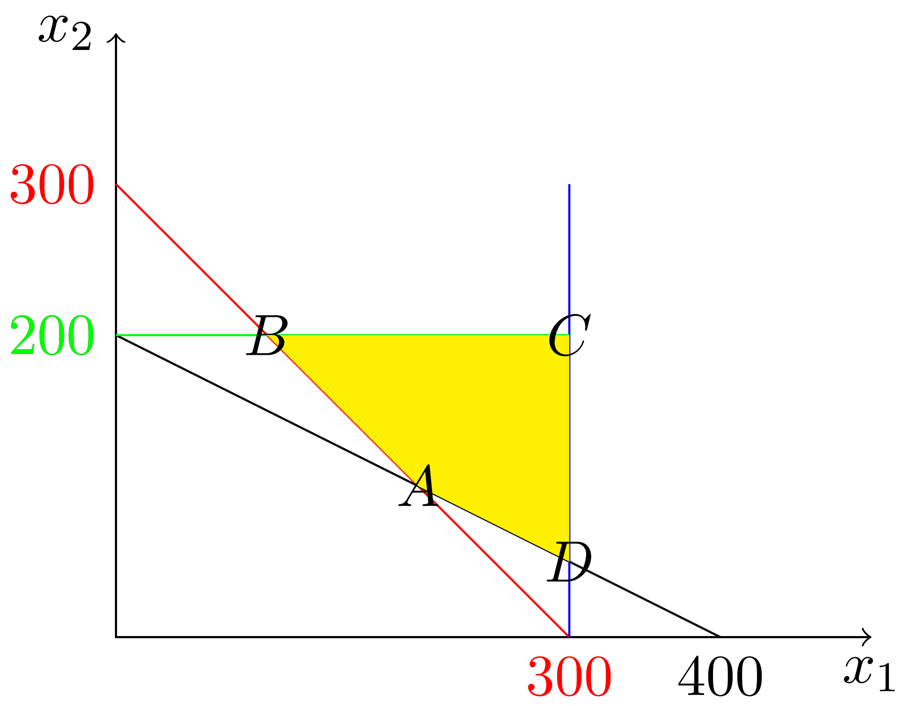
$$x_2 \leq 200 \quad \text{chemical pulping capacity, t/day} \tag{5}$$

$$x_1, x_2 \geq 0. \tag{6}$$

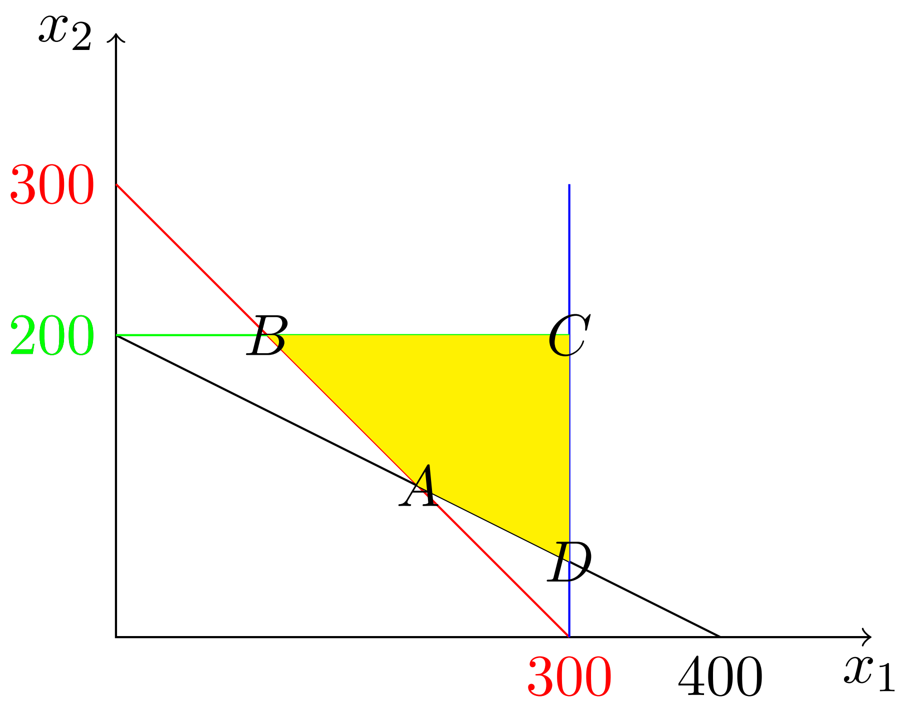
Keeping the river clean - Feasible region and vertices



Keeping the river clean - Feasible region and vertices

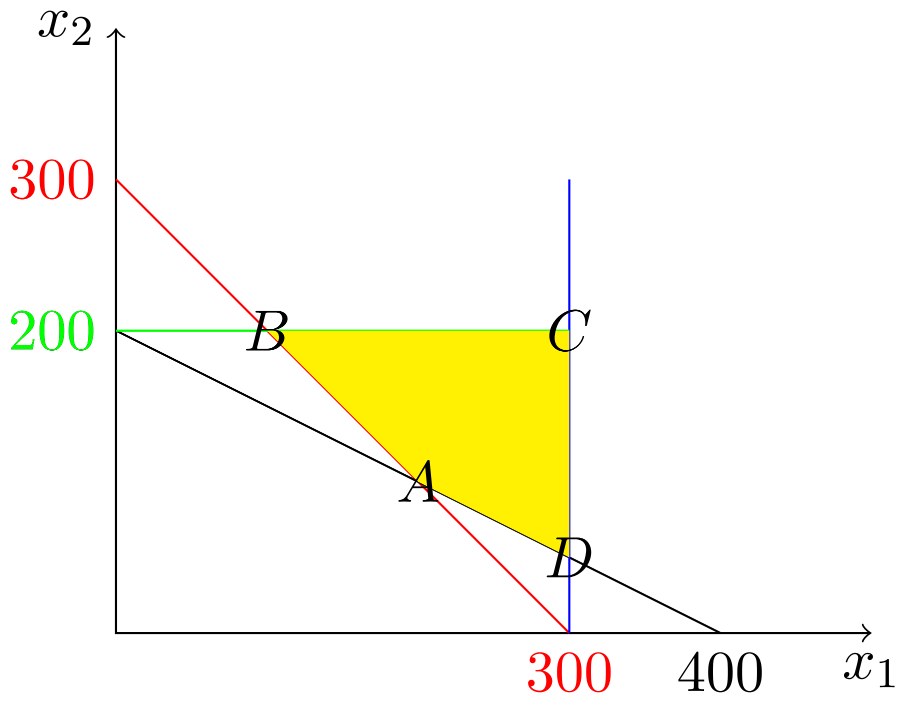


$$A = \begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases} \Leftrightarrow \begin{cases} x_1 = 200 \\ x_2 = 100 \end{cases}$$



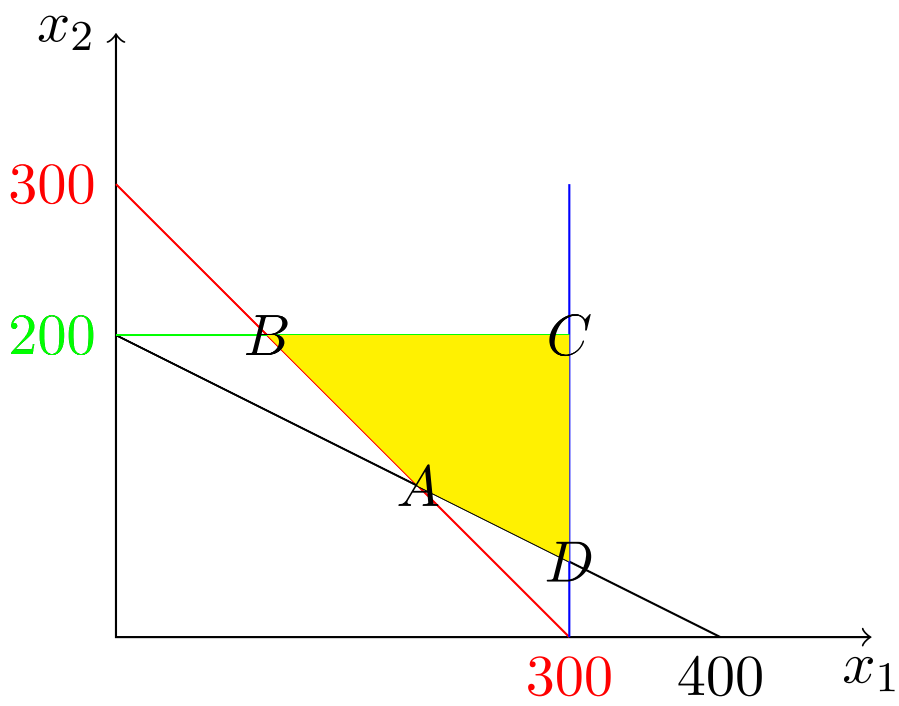
$$B = \begin{cases} x_1 + x_2 = 300 \\ x_2 = 200 \end{cases} \Leftrightarrow \begin{cases} x_1 = 100 \\ x_2 = 200 \end{cases}$$

Keeping the river clean - Feasible region and vertices



$$C = \begin{cases} x_2 = 200 \\ x_1 = 300 \end{cases}$$

Keeping the river clean - Feasible region and vertices



$$D = \begin{cases} x_1 & = & 300 \\ 100x_1 + 200x_2 & = & 40000 \end{cases} \Leftrightarrow \begin{cases} x_1 & = & 300 \\ x_2 & = & 50 \end{cases}$$

For a LPP with k variables.

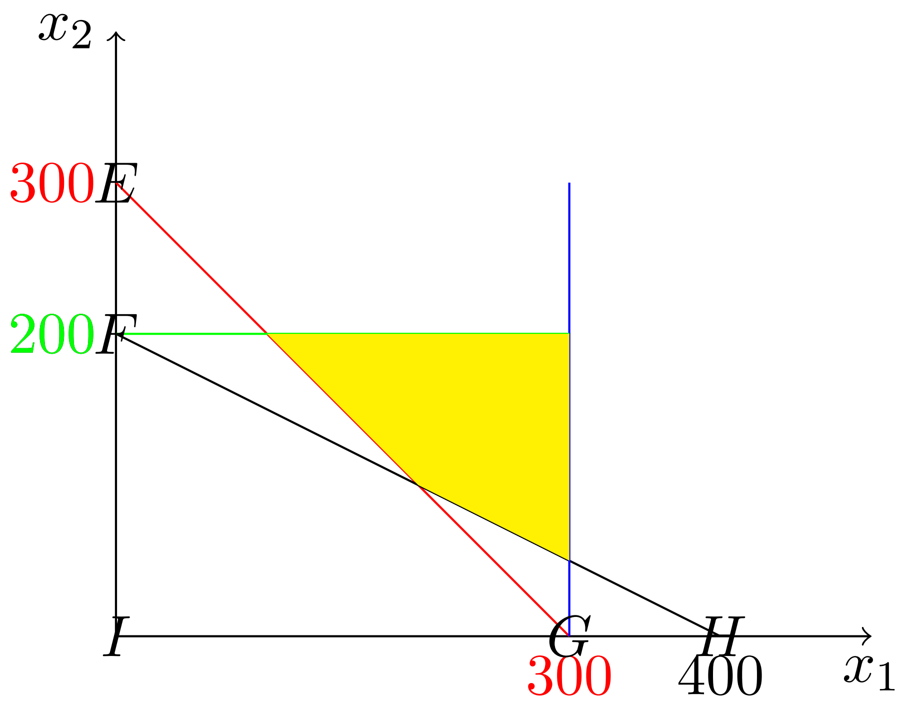
1. A vertex is a feasible solution that is the unique solution of a system with k constraints in the equality.

Example: Keeping the river clean - 2 variables

System	Solution	Violated constraints
$\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	$A(200, 100)$	-
$\begin{cases} x_1 + x_2 = 300 \\ x_2 = 200 \end{cases}$	$B(100, 200)$	-
$\begin{cases} x_2 = 200 \\ x_1 = 300 \end{cases}$	$C(300, 200)$	-
$\begin{cases} x_1 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	$D(300, 50)$	-

It is possible to have systems with k constraints in the equality consistent independent that do not define vertices.

Example: Keeping the river clean - 2 variables



$$E = \begin{cases} x_1 + x_2 = 300 \\ x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 300 \end{cases}$$

Properties of vertices

System	Solution	Violated constraints
$\begin{cases} x_1 + x_2 = 300 \\ x_1 = 0 \end{cases}$	$E(0, 300)$	$x_2 \leq 200$
$\begin{cases} x_1 = 0 \\ x_2 = 200 \end{cases}$	$F(0, 200)$	$x_1 + x_2 \geq 300$
$\begin{cases} x_1 = 0 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	-	-
$\begin{cases} x_2 = 200 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	-	-
$\begin{cases} x_1 = 300 \\ x_2 = 0 \end{cases}$	$G(300, 0)$	$100x_1 + 200x_2 \geq 40000$
$\begin{cases} x_1 = 300 \\ x_1 + x_2 = 300 \end{cases}$	-	-
$\begin{cases} x_1 + x_2 = 300 \\ x_2 = 0 \end{cases}$	-	-

Properties of vertices

The Simplex method at a glance

System	Solution	Violated constraints
$\begin{cases} x_2 = 0 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	$H(400,0)$	$x_1 \leq 300$
$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$	$I(0,0)$	$x_1 + x_2 \geq 300$ $100x_1 + 200x_2 \geq 40000$
$\begin{cases} x_1 = 300 \\ x_1 = 0 \end{cases}$	$\emptyset$	-
$\begin{cases} x_2 = 200 \\ x_2 = 0 \end{cases}$	$\emptyset$	-

$C_2^6 = 15$ systems: 4 are vertices; 9 correspond to unfeasible solutions; 2 are inconsistent.

Can systems with k constraints in the equality be consistent dependent?

Example: Keeping the river clean with one more constraint

$$\min\ Z = x_1 + 1.5x_2 \tag{7}$$
$$x_1 + x_2 \geq 300 \tag{8}$$
$$100x_1 + 100x_2 \geq 30000 \tag{9}$$
$$100x_1 + 200x_2 \geq 40000 \tag{10}$$
$$x_1 \leq 300 \tag{11}$$
$$x_2 \leq 200 \tag{12}$$
$$x_1, x_2 \geq 0. \tag{13}$$

Consistent dependent system	Solution
$\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 100x_2 = 30000 \end{cases}$	$x_1 + x_2 = 300$

Properties of vertices

Consider a LPP with all variables ≥ 0. In this case, if the LPP is feasible, there are vertices!

2. The number of vertices if finite.
The maximum number is C_k^{m+p}, where m is the number of constraints and p is the number of signal constraints.

3. If there is just one optimal solution, this solution is a vertex.

For a LPP with a bounded feasible region

4. If there are alternative solutions, these solutions are vertices and any solution that lies on the straight line segment that joins every pair of those vertices.

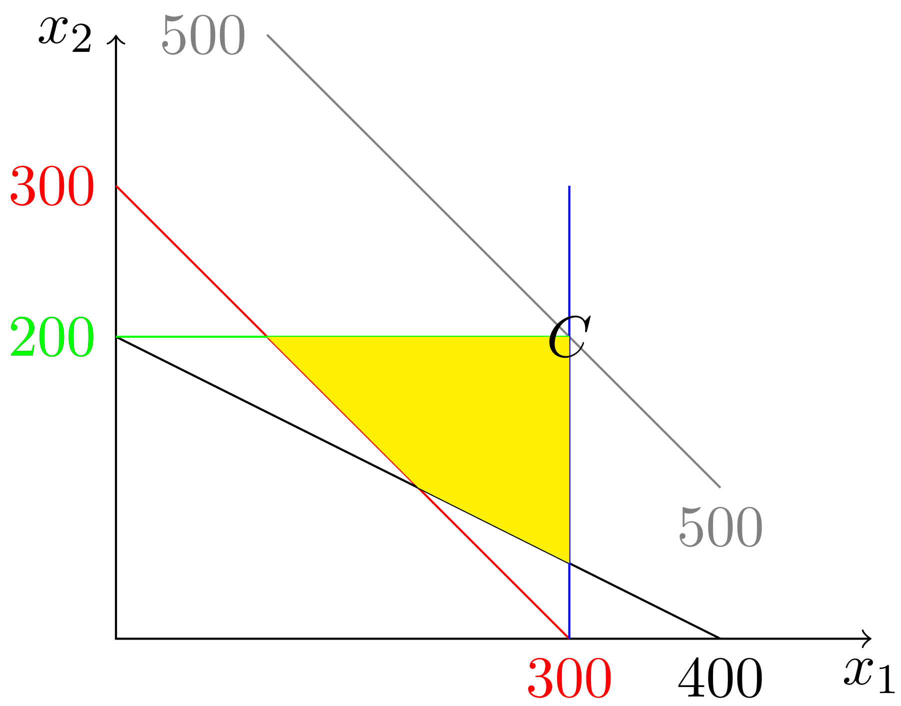
5. If a vertex does not have adjacent vertices with a better objective function then there are no better feasible solutions.

Degenerate vertex

Definition

A degenerate vertex is a feasible solution that is the unique solution of more than one system with k constraints in the equality.

Degenerate vertex



$$C = \begin{cases} x_1 = 300 \\ x_2 = 200 \end{cases} = \begin{cases} x_1 = 300 \\ x_1 + x_2 = 500 \end{cases} = \begin{cases} x_2 = 200 \\ x_1 + x_2 = 500 \end{cases}$$

C is a degenerate vertex.

Definition

LPP is in the standard form if constraints are equations and variables are ≥ 0.

Constraints $\geq$

Example:

$$x_1 + x_2 \geq 300 \Leftrightarrow \begin{cases} x_1 + x_2 - 300 = s_1 \\ s_1 \geq 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x_1 + x_2 - s_1 = 300 \\ s_1 \geq 0 \end{cases}$$

s_1 - Number of workers employed beyond the minimum required.

Generalizing

$$LHS \geq RHS \Leftrightarrow \begin{cases} LHS - s = RHS \\ s \geq 0 \end{cases} .$$

Constraints $\leq$

Example:

$$x_1 \leq 300 \Leftrightarrow \begin{cases} 300 - x_1 = s_3 \\ s_3 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + s_3 = 300 \\ s_3 \geq 0 \end{cases}$$

s_3 - Mechanical pulping capacity that is not used (t/day).

Generalizing

$$LHS \leq RHS \Leftrightarrow \begin{cases} LHS + s = RHS \\ s \geq 0 \end{cases} .$$

Constraints =

Do nothing!

Example: Keeping the river clean

x_1 - Amount of mechanical pulp produced (in tons/day, or t/d)

x_2 - Amount of chemical pulp produced (t/d)

s_1 - Number of workers employed beyond the minimum required

s_2 - Revenue obtained beyond the minimum required (euros/day)

s_3 - Mechanical pulping capacity that is not used (t/day)

s_4 - Chemical pulping capacity that is not used (t/day)

Standard form

$$\min\ Z = x_1 + 1.5x_2 \quad \text{units of BOD per day} \tag{14}$$

$$subject\ to$$

$$x_1 + x_2 - s_1 = 300 \quad \text{workers employed} \tag{15}$$

$$100x_1 + 200x_2 - s2 = 40000 \quad \text{revenue, euros/day} \tag{16}$$

$$x_1 + s_3 = 300 \quad \text{mechanical pulping capacity, t/day} \tag{17}$$

$$x_2 + s_4 = 200 \quad \text{chemical pulping capacity, t/day} \tag{18}$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0. \tag{19}$$

Vertices and feasible basic solutions

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : \left| \begin{array}{lclcl} x_1 & + & x_2 & \geq & 300 \\ 100x_1 & + & 200x_2 & \geq & 40000 \\ x_1 & & & \leq & 300 \\ & & x_2 & \leq & 200 \end{array} \right. , x_1, x_2 \geq 0\}$$

$$\mathcal{F} = \{(x_1, x_2, s_1, s_2, s_3, s_4) \in \mathbb{R}^6 :$$

$$\left| \begin{array}{ccccccccccccc} x_1 & + & x_2 & - & s_1 & & & & & & & = & 300 \\ 100x_1 & + & 200x_2 & & & - & s_2 & & & & & = & 40000 \\ x_1 & & & & & & & + & s_3 & & & = & 300 \\ & & x_2 & & & & & & & + & s_4 & = & 200 \\ x_1, & & x_2, & & s_1, & & s_2, & & s_3, & & s_4 & \geq & 0 \end{array} \right. \}$$

Assumption for further results: No equation (in $\mathcal{F}$) is a direct consequence of the others and thus unnecessary.

Vertices and feasible basic solutions

The Simplex method at a glance

Consider a LPP with all variables ≥ 0, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

Example:

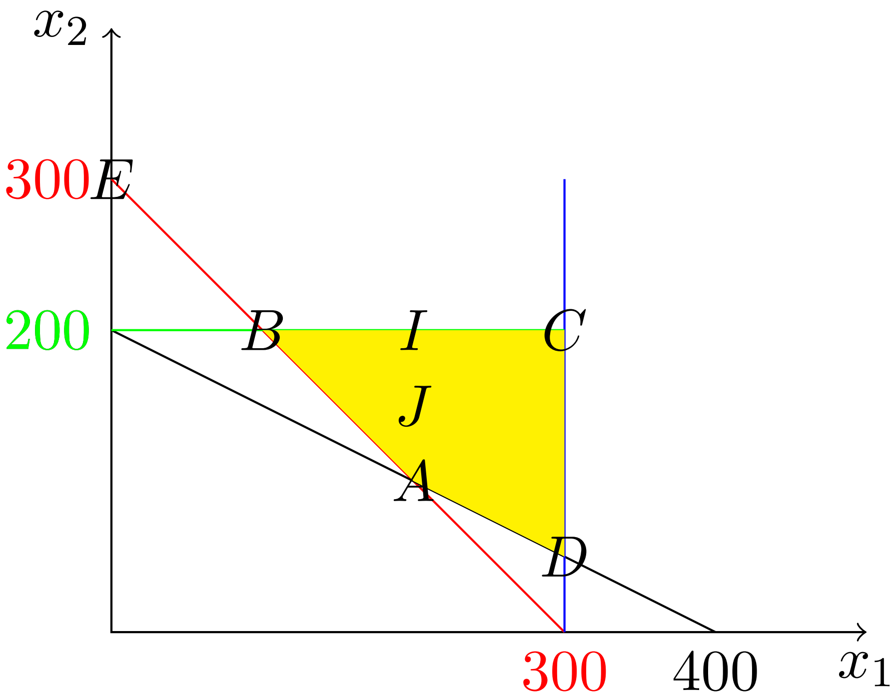
$$(250, 200) \in \mathcal{P} \longrightarrow (250, 200, 150, 25000, 50, 0) \in \mathcal{F}$$

$$(200, 200, 100, 20000, 100, 0) \in \mathcal{F} \longrightarrow (200, 200) \in \mathcal{P}$$

Each vertex in $\mathcal{P}$ corresponds to a **?** in $\mathcal{F}$ and vice-versa.

Vertices and basic feasible solutions

▷ The Simplex method at a glance



Vertices and basic feasible solutions

	(x_1, x_2)	$(x_1, x_2, s_1, s_2, s_3, s_4)$
$A = \begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	$(200, 100)$	(200,100,0,0,100,100)
$B = \begin{cases} x_1 + x_2 = 300 \\ x_2 = 200 \end{cases}$	$(100, 200)$	(100,200,0,10000,200,0)
$C = \begin{cases} x_2 = 200 \\ x_1 = 300 \end{cases}$	$(300, 200)$	(300,200,200,30000,0,0)
$D = \begin{cases} x_1 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$	$(300, 50)$	(300,50,50,0,0,150)
I	$(200, 200)$	(200,200,100,20000,100,0)
J	$(200, 150)$	(200,150,50,10000,100,50)
$E = \begin{cases} x_1 = 0 \\ x_1 + x_2 = 300 \end{cases}$	$(0, 300)$	(0,300,0,20000,300,-100)

Vertices and basic feasible solutions

	(x_1, x_2)	$(x_1, x_2, s_1, s_2, s_3.s_4)$
A	$(200, 100)$	(200,100,0,0,100,100)
B	$(100, 200)$	(100,200,0,10000,200,0)
C	$(300, 200)$	(300,200,200,30000,0,0)
D	$(300, 50)$	(300,50,50,0,0,150)
I	$(200, 200)$	(200,200,100,20000,100,0)
J	$(200, 150)$	(200,150,50,10000,100,50)
E	$(0, 300)$	(0,300,0,20000,300,-100)

Vertices and basic feasible solutions

Each vertex in $\mathcal{P}$ corresponds to a basic feasible solution in $\mathcal{F}$ and vice-versa.

Definition

A basic feasible solution (in $\mathcal{F}$, with a system with n variables and m equations):

- has all components non-negative
- has at least $n - m$ components equal to zero
- is the unique solution of the system of equations setting the variables associated to these components equal to zero.

"is the unique solution of the system of equations setting the variables associated to these components equal to zero" is really necessary? Yes!

Example:

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : \left| \begin{array}{lllll} x_1 & + & x_2 & \geq & 300 \\ 100x_1 & + & 200x_2 & \geq & 40000 \\ x_1 & & & \leq & 300 \\ & & x_2 & \leq & 200 \\ 2x_1 & + & 2x_2 & \geq & 600 \end{array} \right. , x_1, x_2 \geq 0\}$$

$$\mathcal{F} = \{(x_1, x_2, s_1, s_2, s_3, s_4, s_5) \in \mathbb{R}^7 : \left| \begin{array}{ccccccccccccccc} x_1 & + & x_2 & - & s_1 & & & & & & & & & = & 300 \\ 100x_1 & + & 200x_2 & & & - & s_2 & & & & & & & = & 40000 \\ x_1 & & & & & & & + & s_3 & & & & & = & 300 \\ & & x_2 & & & & & & & + & s_4 & & & = & 200 \\ 2x1 & + & 2x_2 & & & & & & & & & - & s_5 & = & 600 \\ x_1, & & x_2, & & s_1, & & s_2, & & s_3, & & s_4, & & s_5 & \geq & 0 \end{array} \right. \}$$

No equation (in $\mathcal{F}$) is a direct consequence of the others.

Vertices and basic feasible solutions

The Simplex method at a glance

$(150, 150)$ corresponds to $(150, 150, 0, 5000, 150, 50, 0)$ which

$\longrightarrow$ has all components non-negative

$\longrightarrow$ has (at least) $7 - 5 = 2$ components equal to zero

$\longrightarrow$ but, it is not the unique solution of the system of equations setting the variables associated to these components equal to zero.

$$\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 - s_2 = 40000 \\ x_1 + s_3 = 300 \\ x_2 + s_4 = 200 \\ 2x1 + 2x_2 = 600 \end{cases}$$

is consistent dependent.

$(150, 150, 0, 5000, 150, 50, 0)$ is not a feasible basic solution and $(150, 150)$ is not a vertex!

Degenerate basic feasible solution

Definition

A basic feasible solution is degenerate if it has more than $n - m$ null components.

A degenerate basic feasible solution corresponds to a degenerate vertex and vice-versa.

Simplex method at a glance

Convert the linear programming model to the standard form

Find an initial feasible basic solution. If there is no feasible basic solution, the problem is unfeasible and STOP

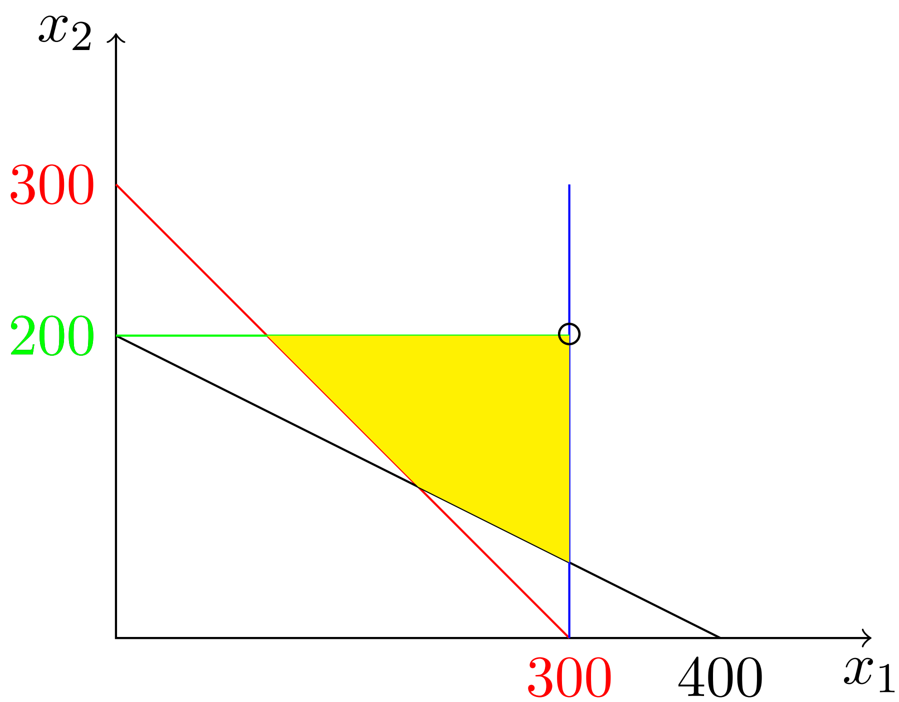
repeat

Verify if the objective function value can be improved. If not, an optimal solution was found and STOP

Move to the adjacent feasible basic solution in the direction that most improves the objective function. If there is no such solution, the problem is unbounded! STOP

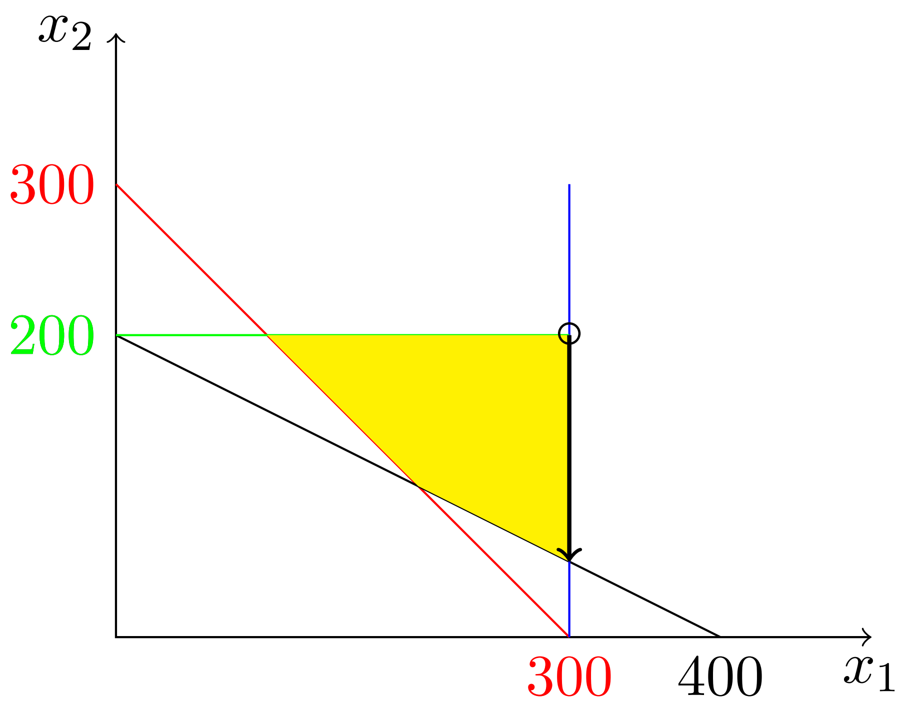
until Stopping criteria is fulfilled

▷ The Simplex method at a glance

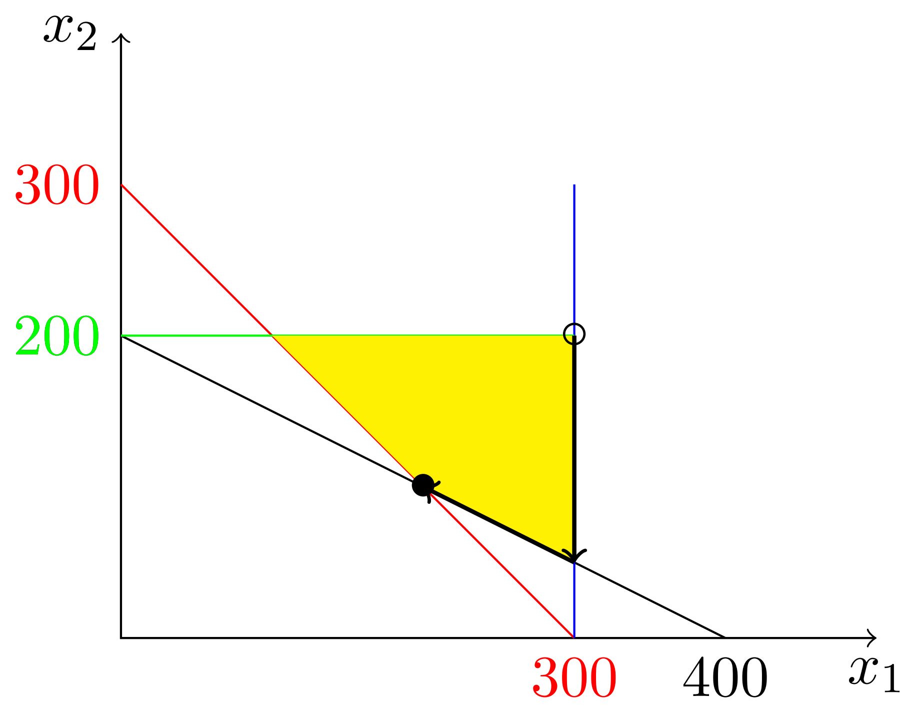


$$\min\ Z = x_1 + 1.5x_2$$

▷ The Simplex method at a glance



$$\min\ Z = x_1 + 1.5x_2$$



$$\min\ Z = x_1 + 1.5x_2$$