The Simplex method at a glance (conclusion)

8 Mar 2019

Simplex method at a glance

Convert the linear programming model to the standard form

Find an initial feasible basic solution. If there is no feasible basic solution, the problem is unfeasible and STOP

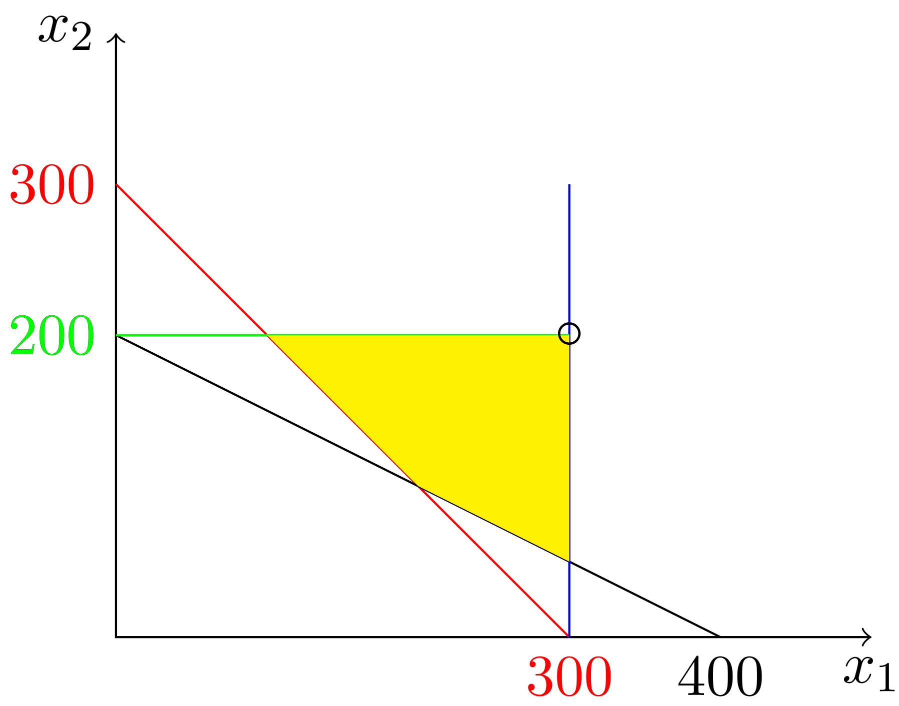
repeat

Verify if the objective function value can be improved. If not, an optimal solution was found and STOP

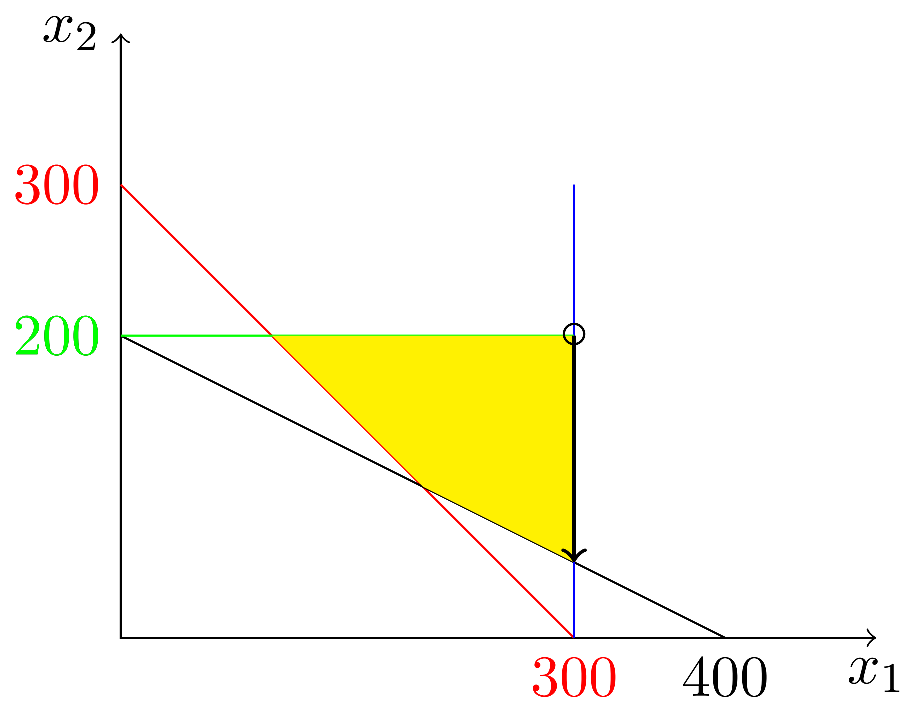
Move to the adjacent feasible basic solution in the direction that most improves the objective function. If there is no such solution, the problem is unbounded! STOP

until Stopping criteria is fulfilled

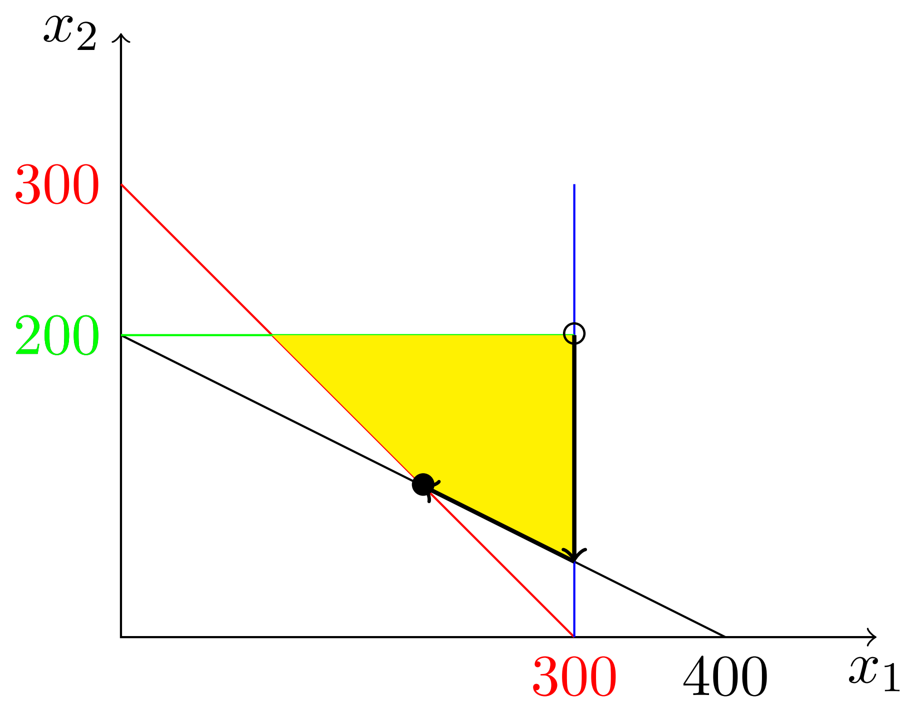
Simplex method at a glance



$$\min\ Z = x_1 + 1.5x_2$$



$$\min Z = x_1 + 1.5x_2$$



$$\min\ Z = x_1 + 1.5x_2$$

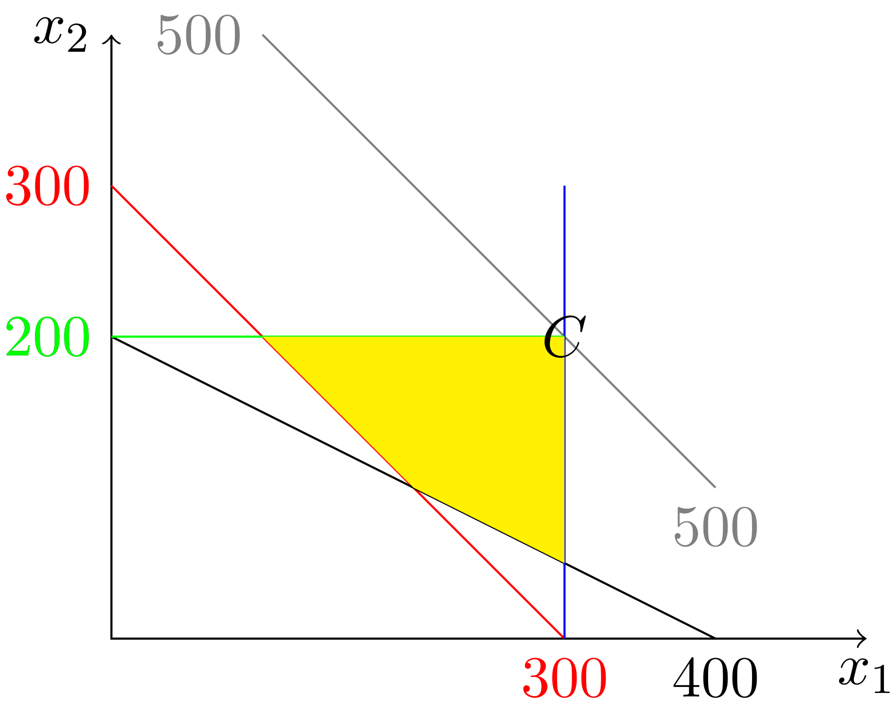
Basic variables and non-basic variables

n - number of variables in the standard form

m - number of equations in the standard form

- Each basic feasible solution has $n - m$ null components and m positive components or, if it is degenerate, more than $n - m$ null components and less than m positive components.
- Each basic feasible solution has m **basic variables** - those associated to the positive components and eventually to null components if the solution has less than m positive components. The other $n - m$ variables are the so-called **non-basic variables**.
- Each basic feasible solution is the unique solution of the system of linear equations with the non-basic variables equal to zero.

Example of a degenerate basic feasible solution



▷ Simplex method at a glance
Coming back to sensitivity analysis

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : \left| \begin{array}{lcll} x_1 & + & x_2 & \geq & 300 \\ 100x_1 & + & 200x_2 & \geq & 40000 \\ x_1 & & & \leq & 300 \\ & & x_2 & \leq & 200 \\ x_1 & + & x_2 & \leq & 500 \end{array} \right. , x_1, x_2 \geq 0\}$$

$$\mathcal{F} = \{(x_1, x_2, s_1, s_2, s_3, s_4, s_5) \in \mathbb{R}^7 : \left| \begin{array}{ccccccccccccccc} x_1 & + & x_2 & - & s_1 & & & & & & & & & = & 300 \\ 100x_1 & + & 200x_2 & & & - & s_2 & & & & & & & = & 40000 \\ x_1 & & & & & & & + & s_3 & & & & & = & 300 \\ & & x_2 & & & & & & & + & s_4 & & & = & 200 \\ x1 & + & x_2 & & & & & & & & & + & s_5 & = & 500 \\ x_1, & & x_2, & & s_1, & & s_2, & & s_3, & & s_4, & & s_5 & \geq & 0 \end{array} \right. \}$$

No equation (in $\mathcal{F}$) is a direct consequence of the others and thus unnecessary.

Basic and non-basic variables

$\longrightarrow$ Each basic feasible solution has at least $7 - 5 = 2$ null components.

$\longrightarrow$ $C = (300, 200)$ corresponds to $(300, 200, 200, 30000, 0, 0, 0)$.

$\longrightarrow$ The basic variables are x_1, x_2, s_1, s_2 and one of the other variables such that the basic feasible solution is the unique solution of the system of linear equations with the non-basic variables equal to zero (s_3, s_4 or s_5 can be a basic variable).

Basic and non-basic variables

Definition

Two basic feasible solutions are adjacent if they have all non-basic variables (basic variables) in common except one.

Example:

$A = (200, 100) \longleftrightarrow (200, 100, 0, 0, 100, 100)$ with the non-basic variables s_1, s_2 and the basic variables x_1, x_2, s_3, s_4

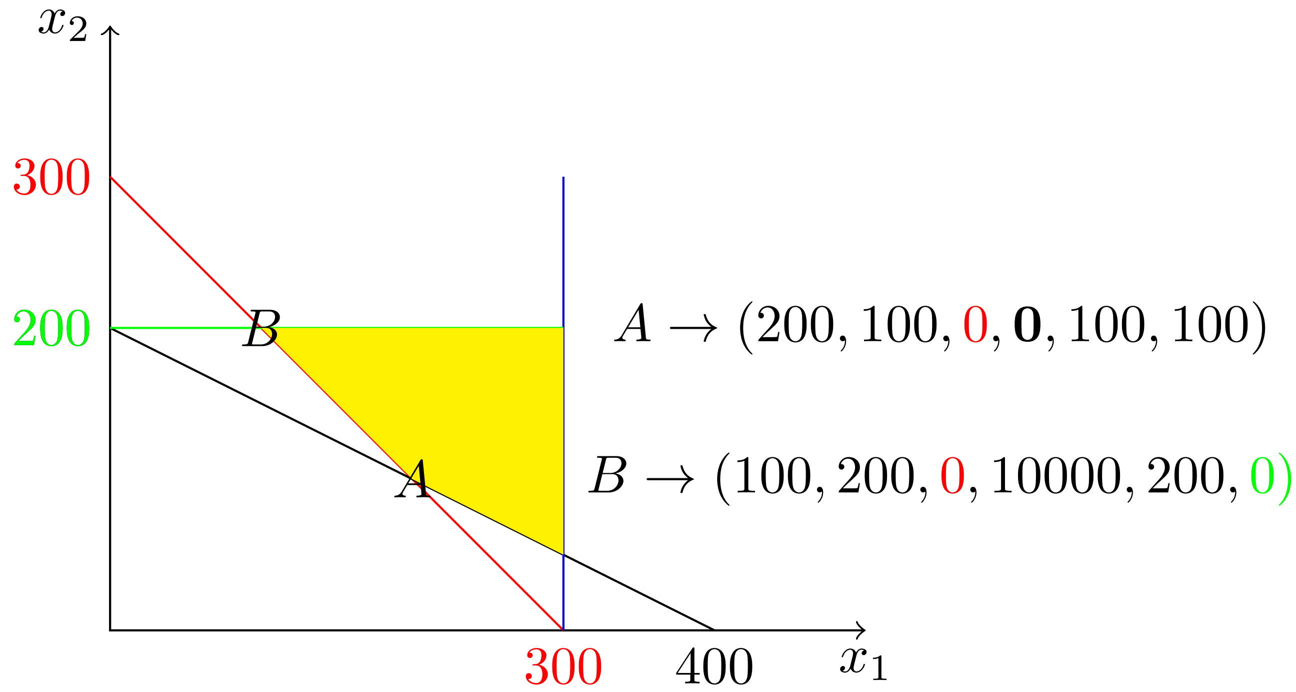
$B = (100, 200) \longleftrightarrow (100, 200, 0, 10000, 200, 0)$ with the non-basic variables s_1, s_4 and the basic variables x_1, x_2, s_2, s_3

$(200, 100, 0, 0, 100, 100)$ and $(100, 200, 0, 10000, 200, 0)$ have in common all non-basic variables (basic variables) except one

$(200, 100, 0, 0, 100, 100)$ and $(100, 200, 0, 10000, 200, 0)$ are adjacent basic feasible solutions.

- Two adjacent basic feasible solutions correspond to two adjacent vertices.

The defining-equations of two adjacent vertices are equal except in one equation. Example:



The Simplex method at a glance

- Starting at $(300, 200, 200, 30000, 0, 0)$.

$$\begin{cases} Z = x_1 + 1.5x_2 \\ \\ x_1 + x_2 - s_1 = 300 \\ 100x_1 + 200x_2 - s_2 = 40000 \\ x_1 + s_3 = 300 \\ x_2 + s_4 = 200 \end{cases}$$

- Search for a better basic feasible solution

$$\begin{cases} Z = 600 - s_3 - 1.5s_4 & & \\ \\ x_1 = 300 - s_3 & \Delta s_3 = 300 & \\ x_2 = 200 - s_4 & & \\ s_1 = 200 - s_3 - s_4 & \mathbf{\Delta s_3 = 200} & \mathbf{\Delta Z = -200} \\ s_2 = 30000 - 100s_3 - 200s_4 & \Delta s_3 = 300 & \end{cases}$$

$$\begin{cases} Z = 600 - s_3 - 1.5s_4 & & \\ \\ x_1 = 300 - s_3 & & \\ x_2 = 200 - s_4 & \Delta s_4 = 200 & \\ s_1 = 200 - s_3 - s_4 & \Delta s_4 = 200 & \\ s_2 = 30000 - 100s_3 - 200s_4 & \mathbf{\Delta s_4 = 150} & \mathbf{\Delta Z = -225} \end{cases}$$

$$\begin{cases} \mathbf{Z} = 600 - 1.5(150) = \mathbf{375} \\ \\ x_1 = 300 \\ x_2 = 200 - 150 = 50 \\ s_1 = 200 - 150 = 50 \\ \mathbf{s_4} = \mathbf{150} \end{cases} \longrightarrow (300, 50, 50, \mathbf{0}, 0, \mathbf{150}) \longrightarrow D(300, 50)$$

- Search for a better basic feasible solution

$$\begin{cases} Z = 375 + \frac{3}{400}s_2 - \frac{1}{4}s_3 & \\ x_1 = 300 - s_3 & \Delta s_3 = 300 \\ x_2 = 50 + \frac{1}{200}s_2 + \frac{1}{2}s_3 & \\ s_1 = 50 + \frac{1}{200}s_2 - \frac{1}{2}s_3 & \boldsymbol{\Delta s_3 = 100} \qquad \boldsymbol{\Delta Z = -25} \\ s_4 = 150 - \frac{1}{200}s_2 - \frac{1}{2}s_3 & \Delta s_3 = 300 \end{cases}$$

$$\begin{cases} Z = 375 - \frac{1}{4}(100) = 350 \\ x_1 = 200 \\ x_2 = 50 + \frac{1}{2}(100) = 100 \\ \mathbf{s_3} = \mathbf{100} \\ s_4 = 150 - \frac{1}{2}(100) = 100 \end{cases} \longrightarrow (200, 100, \mathbf{0}, 0, \mathbf{100}, 100) \longrightarrow A(200, 100)$$

- Search for a better basic feasible solution

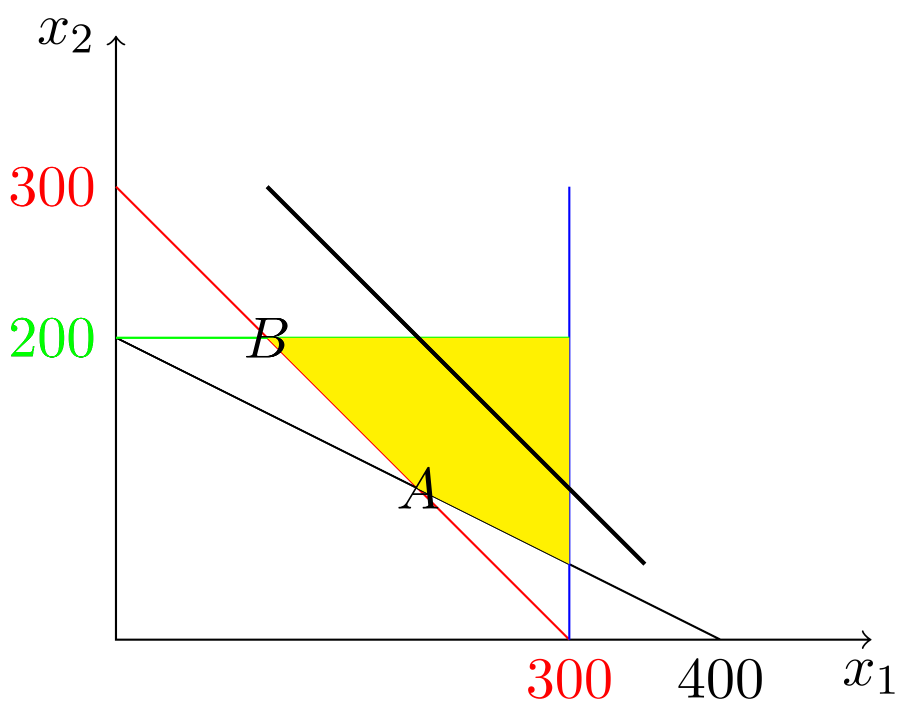
$$\begin{cases} Z &= 350 + \frac{1}{2}s_1 + \frac{1}{400}s_2 \\ x_1 &= 200 + 2s_1 - \frac{1}{100}s_2 \\ x_2 &= 100 - s_1 + \frac{1}{100}s_2 \\ s_3 &= 100 - 2s_1 - \frac{1}{100}s_2 \\ s_4 &= 100 + s_1 - \frac{1}{100}s_2 \end{cases}$$

The objective function value can not be improved. $A(200, 100)$ is an optimal solution!

The Simplex method at a glance - optimal alternative solutions

Keeping the rive clean with the objective function

$$\text{Min } Z = x_1 + x_2$$



- Starting at $(300, 200, 200, 30000, \color{blue}{0}, \color{green}{0})$.

$$\begin{cases} Z = x_1 + x_2 \\ \\ x_1 + x_2 - s_1 = 300 \\ 100x_1 + 200x_2 - s_2 = 40000 \\ x_1 + s_3 = 300 \\ x_2 + s_4 = 200 \end{cases}$$

- Search for a better basic feasible solution

$$\begin{cases} Z = 500 - s_3 - s_4 \\ \\ x_1 = 300 - s_3 & \Delta s_3 = 300 \\ x_2 = 200 - s_4 \\ s_1 = 200 - s_3 - s_4 & \mathbf{\Delta s_3 = 200} & \mathbf{\Delta Z = -200} \\ s_2 = 30000 - 100s_3 - 200s_4 & \Delta s_3 = 300 \end{cases}$$

$$\begin{cases} Z = 500 - s_3 - s_4 \\ \\ x_1 = 300 - s_3 \\ x_2 = 200 - s_4 & \Delta s_4 = 200 \\ s_1 = 200 - s_3 - s_4 & \Delta s_4 = 200 \\ s_2 = 30000 - 100s_3 - 200s_4 & \mathbf{\Delta s_4 = 150} & \mathbf{\Delta Z = -150} \end{cases}$$

$$\begin{cases} \mathbf{Z} = 500 - 1(200) = \mathbf{300} \\ \\ x_1 = 300 - 200 = 100 & \longrightarrow \\ x_2 = 200 \\ s_2 = 30000 - 20000 = 10000 \\ \mathbf{s_3} = \mathbf{200} \end{cases}$$

$(100, 200, \mathbf{0}, 10000, \mathbf{200}, 0) \longrightarrow B(100, 200)$

The Simplex method at a glance

- Search for a better basic feasible solution

$$\begin{cases} Z = 300 + s_1 + 0s_4 \\ x_1 = 100 + s_1 + s_4 \\ x_2 = 200 - s_4 & \Delta s_4 = 200 \\ s_3 = 200 - s_1 - s_4 & \Delta s_4 = 200 \\ s_2 = 10000 + 100s_1 - 100s_4 & \mathbf{\Delta s_4 = 100} & \mathbf{\Delta Z = 0(100) = 0} \end{cases}$$

$$\begin{cases} \mathbf{Z} = 300 + 0(100) = \mathbf{300} \\ x_1 = 100 + 100 = 200 \\ x_2 = 200 - 100 = 100 \\ s_3 = 200 - 100 = 100 \\ \mathbf{s_4} = \mathbf{100} \end{cases} \longrightarrow (200, 100, \mathbf{0}, 0, 100, \mathbf{100}) \longrightarrow A(200, 100)$$

- A has the same objective function value than B. So, there is no vertex adjacent to B better than B. Thus, A and B are optimal solutions (alternative solutions).

Reduced cost

Definition

The **reduced cost** of a non-basic variable, which has zero value in the optimal solution, provides a measure of how much the objective function would change (a penalty amount) if one unit of this variable was forced into the solution.

A basic variable has a null reduced cost.