Applied Operations Research

Solving applications of linear programming with Excel

Question 1.

Land use planning
A river is a major source of water of a certain city, and the council has an annual plan to expand the city's development in the area along the river. A total of 100 acres of land is projected to be needed for residential, business, and recreational use. According to the plan, at least 20 acres of land should be designed to for residential development, 30 acres will be used for business development, and a recreational park will be built on at least 10 acres. The initial investment cost for residential land is 8 million euros for the first 20 acres of land and 300000 euros for extra acre of land thereafter. The initial investment costs for business land and recreational land are 20 million and 12 million euros, respectively, while the costs for additional land are 500000 and 400000 euros per acre, respectively. An acre of residential land can yield a profit of 50000 euros per year, and the expected annual profits for business land and recreational land are 120000 and 150000 euros, respectively. On average, every acre of residential land will use 20 m^3 of water per month, and every acre of business land and recreational land consumes 40 m^3 and 25 m^3 of water per month, respectively. The annual budget is 80 million euros, and the regulation of water from the river is 40000 m^3 per year. The city council wants to find an annual plan with the maximum profit.

1. **Formulate this problem as a LP model.**

   The decision variables are as follows:

   x_1 - area to residential development (acre)
   x_2 - area to business development (acre)
   x_3 - area to recreational development (acre)

   The model is as follows:

$$\max\ z = 50000x_1 + 120000x_2 + 150000x_3 \tag{1}$$
subject to
$$x_1 + x_2 + x_3 = 100 \tag{2}$$
$$x_1 \geq 20 \tag{3}$$
$$x_2 \geq 30 \tag{4}$$
$$x_3 \geq 10 \tag{5}$$
$$300000x_1 + 500000x_2 + 400000x_3 \leq 65000000^{\star} \tag{6}$$
$$240x_1 + 480x_2 + 300x_3 \leq 40000 \tag{7}$$
$$x_1, x_2, x_3 \geq 0. \tag{8}$$

$$\star\ \overbrace{8000000 + 300000(x_1 - 20)}^{\text{residential}} + \overbrace{20000000 + 500000(x_2 - 30)}^{\text{business}} + \overbrace{12000000 + 400000(x_3 - 10)}^{\text{recreational}} \leq 80000000$$

   Expression (1) maximizes the profit. Constraint (2) ensures that the whole land is exploited. Constraints (3), (4) and (5) guarantee that the minimum area for each development (residential, business and recreational) is exploited. Constraint (6) is the annual budget constraint. Constraint (7) is the annual regulation of water constraint. Constraints (8) state the non-negativity requirements on the variables.

2. **Solve the model with the Excel Solver. Which constraints are binding at the optimal solution found (saturated constraints)? Comment.**

   The optimal solution found is $x_1^* = 20$ acre, $x_2^* = 30$ acre and $x_3^* = 50$ acre with the annual profit of $Z^* = 12100000$ €. Constraints (2), (3) and (4) are binding. In other words, the total area is used as well as the minimum areas for residential and business developments. With respect to the area for recreational development, 40 acre more than the minimum required are used. With respect to the budget and regulation of water, 24000000 € and 5800 m^3, respectively, are not spent.

3. **Report the shadow prices for each constraint and comment.**

   The shadow price of a constraint measures the impact on the optimal objective value with the (slight) increase of the RHS, remaining the other parameters the same.

   The shadow prices of the non-binding constraints (constraints (5), (6) and (7)) are equal to zero. The shadow prices of the binding constraints are: 150000 €/acre for constraint (2), -100000 €/acre for constraint (3) and -30000 €/acre for constraint (4).

   In other words, a slight increase on the minimum area for recreational development, budget and regulation of water do not have impact on the optimal annual profit. The increase on the total area has a positive impact of 150000 €/acre. The increase on the minimum areas for residential or business development have a negative impact of 100000 €/acre and 30000 €/acre, respectively.

4. **Derive the range of feasibility for all RHS values.**

   The shadow prices are valid for RHS increases by, at most, 19.33 acre (constraint (2)), 40 acre (constraint (3)), 32.22 acre (constraint (4)), 40 acre (constraint (5)) and $+\infty$ (constraints (6) and (7)). Thus,

   i) if the total area increased from 100 acre to 119.33 acre, the optimal profit would increase 150000 times 19.33, that is, 2900000 €
   ii) if the minimum area for residential development increased from 20 acre to 60 acre, the optimal profit would decrease 100000 times 40, that is, 4000000 €
   iii) if the minimum area for business development increased from 30 acre to 62.22 acre, the optimal profit would decrease 30000 times 32.22, that is, 966666.67 €
   iv) if the minimum area for recreational development increased from 10 acre to 50 acre, the optimal profit would not change
   v) if the budget and the regulation of water increased infinitely, the optimal profit would not change.

   The shadow prices are valid for RHS decreases by, at most, 40 acre (constraint (2)), 20 acre (constraint (3)), 30 acre (constraint (4)), $-\infty$ (constraint (5)), 24000000 (constraints (6)) and 5800 (constraints (7)). Thus,

   i) if the total area decreased from 100 acre to 60 acre, the optimal profit would decrease 150000 times 40, that is, 6000000 €
   ii) if the minimum area for residential development decreased from 20 acre to 0 acre, the optimal profit would increase 100000 times 20, that is, 2000000 €
   iii) if the minimum area for business development decreased from 30 acre to 0 acre, the optimal profit would increase 30000 times 30, that is, 900000 €
   iv) if the minimum area for recreational development decreased infinitely (in fact to zero), the optimal profit would not change
   v) if the budget decreased from 80000000 € to 56000000 €, the optimal profit would not change
   vi) if the regulation of water decreased from 40000 m^3 to 34200 m^3, the optimal profit would not change.

5. Determine an expression that gives the optimal objective values for the RHS values mentioned in d) for each constraint.

   For constraints

   (2): $y = 12100000 + 150000(x - 100),\ x \in [60, 119.33]$

   (3): $y = 12100000 - 100000(x - 20),\ x \in [0, 60]$

   (4): $y = 12100000 - 30000(x - 30),\ x \in [0, 62.22]$

   (5): $y = 12100000,\ x \in [0, 50]\ (] - \infty, 50])$

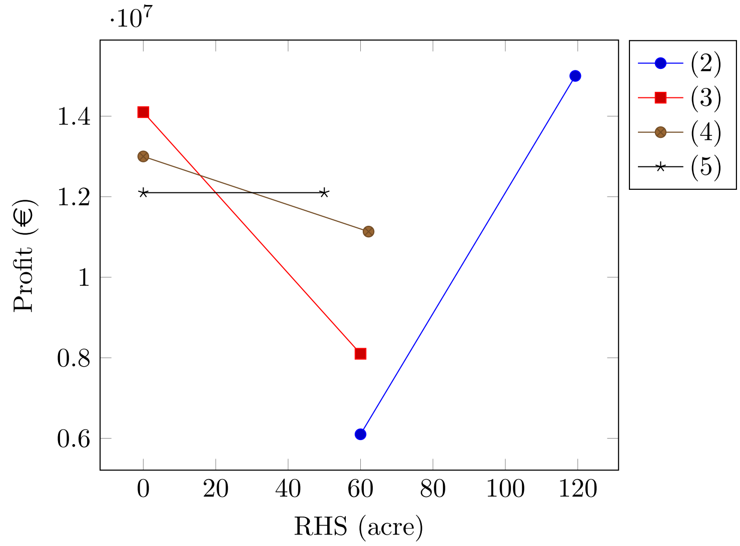

Figure 1: Optimal objective value according to the RHS of constraints (2), (3), (4) and (5). The slope of each line segment is the shadow price of the respective constraint.

(6): $y = 12100000$, $x \in [56000000, +\infty[$

(7): $y = 12100000$, $x \in [34200, +\infty[$,

where y is the optimal annual profit in euros and x is the RHS of each constraint. The slope of each line segment is the shadow price (Figure 1).

6. **Derive the range of optimality for all objective function coefficients and comment.**

   Columns "Allowable Increase" and "Allowable Decrease" give the amount by which each objective function coefficient (in column "Objective Coefficient") can be increased or decreased, respectively, without changing the optimal activity levels.

   Thus, as long as the values of the objective function coefficients on x_1 (annual residential profit per acre), x_2 (annual business profit per acre) and x_3 (annual recreational profit per acre) are, respectively, in the following intervals, one at a time and remaining the other parameters unchanged, the optimal solution will be the same ($x_1^* = 20$ $x_2^* = 30$ and $x_3^* = 50$):

   i) $]-\infty, 150000]$ (the allowable decrease is $+\infty$ and the allowable increase is 100000)
   ii) $]-\infty, 150000]$ (the allowable decrease is $+\infty$ and the allowable increase is 30000)
   iii) $[120000, +\infty[$ (the allowable decrease is 30000 and the allowable increase is $+\infty$).

7. **Solve the problem without using the simplex method.**

   As $x_3 = 100 - x_2 - x_1$ (constraints (2)), the model can be rewritten as

$$\max\ z = 50000x_1 + 120000x_2 + 150000(100 - x_2 - x_1) \tag{9}$$
$$\text{subject to}$$
$$x_1 \geq 20 \tag{10}$$
$$x_2 \geq 30 \tag{11}$$
$$100 - x_2 - x_1 \geq 10 \tag{12}$$
$$300000x_1 + 500000x_2 + 400000(100 - x_2 - x_1) \leq 65000000 \tag{13}$$
$$240x_1 + 480x_2 + 300(100 - x_2 - x_1) \leq 40000 \tag{14}$$
$$x_1, x_2 \geq 0 \tag{15}$$

or, equivalently

$$
\begin{aligned}
&\max\ z = -100000x_1 - 30000x_2 + 15000000 && (16)\\
&\text{subject to}\\
&x_1 \geq 20 && (17)\\
&x_2 \geq 30 && (18)\\
&x_2 + x_1 \leq 90 && (19)\\
&-100000x_1 + 100000x_2 \leq 25000000 && (20)\\
&-60x_1 + 180x_2 \leq 10000 && (21)\\
&x_1, x_2 \geq 0. && (22)
\end{aligned}
$$

The optimal solution of this problem is $x_1^* = 20$ and $x_2^* = 30$ - notice that positive values for these variables will decrease the objective function value!

Thus, $x_3^* = 100 - x_2^* - x_1^* = 50$ acre.

Question 2.

Fishery planning

A fishery has about 120 tons of fish, including 30 tonnes of salmon, 50 tonnes of tuna, and 40 tonnes of sardines. Every year, fishermen capture a certain amount of fish and sell them to the market. Assume that the average market price for salmon was 6 € per Kg, for tuna was 5.50 € per Kg, and for sardines was 5 € per Kg. To maintain an ecological equilibrium, the fishery manager would like to keep the amount of tuna less than twice the amount of salmon every year. Suppose the overall average reproduction rate of these three fish is 12% per year. The fishery manager would like to keep the expected amount of fish in the fishery after four years to still be at least 120 tonnes. Additionally, the fishery manager would like to have at least 12 tonnes of salmon, 25 tonnes of tuna, and 16 tonnes of sardines in the fishery at the end of the four years. Find a fishery pattern so that fishermen can maximize their present profit over the 4-year period. Assume that capturing would be made at the end of each year and the annual interest rate is 4%.

1. **Formulate this problem as a LP model.**

   The decision variables are as follows:

   x_{it} - amount of fish (in ton) of species i captured at the end of year t, $i = 1$ (salmon), $i = 2$ (tuna), $i = 3$ (sardine), $t = 1, 2, 3, 4$

   S_{it} - amount of fish (in ton) of species i in the fishery at the end of year t, $i = 1.2.3$, $j = 1, 2, 3, 4$ (auxiliary variables).

   The present profit p_{it} (in €) obtained per tonne of fish captured, according to species i and year t, is as follows:

	Present profit p_{it} (€/ton)			
	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$i = 1$	5769.23	5547.34	5333.98	5128.83
$i = 2$	5288.46	5085.06	4889.48	4701.42
$i = 3$	4807.69	4622.78	4444.98	4274.02

   The model is as follows:

$$\max z = \sum_{i=1}^{3} \sum_{t=1}^{4} p_{it} x_{it} \tag{23}$$

$$subject\,to$$

$$x_{11} \leq 1.12 \times 30 \tag{24}$$

$$x_{21} \leq 1.12 \times 50 \tag{25}$$

$$x_{31} \leq 1.12 \times 40 \tag{26}$$

$$S_{11} = 1.12 \times 30 - x_{11} \tag{27}$$

$$S_{21} = 1.12 \times 50 - x_{21} \tag{28}$$

$$S_{31} = 1.12 \times 40 - x_{31} \tag{29}$$

$$x_{12} \leq 1.12 S_{11} \tag{30}$$

$$x_{22} \leq 1.12 S_{21} \tag{31}$$

$$x_{32} \leq 1.12 S_{31} \tag{32}$$

$$S_{12} = 1.12 S_{11} - x_{12} \tag{33}$$

$$S_{22} = 1.12 S_{21} - x_{22} \tag{34}$$

$$S_{32} = 1.12 S_{31} - x_{32} \tag{35}$$

$$x_{13} \leq 1.12 S_{12} \tag{36}$$
$$x_{23} \leq 1.12 S_{22} \tag{37}$$
$$x_{33} \leq 1.12 S_{32} \tag{38}$$
$$S_{13} = 1.12 S_{12} - x_{13} \tag{39}$$
$$S_{23} = 1.12 S_{22} - x_{23} \tag{40}$$
$$S_{33} = 1.12 S_{32} - x_{33} \tag{41}$$
$$x_{14} \leq 1.12 S_{13} \tag{42}$$
$$x_{24} \leq 1.12 S_{23} \tag{43}$$
$$x_{34} \leq 1.12 S_{33} \tag{44}$$
$$S_{14} = 1.12 S_{13} - x_{14} \tag{45}$$
$$S_{24} = 1.12 S_{23} - x_{24} \tag{46}$$
$$S_{34} = 1.12 S_{33} - x_{34} \tag{47}$$
$$S_{21} \leq 2 S_{11} \tag{48}$$
$$S_{22} \leq 2 S_{12} \tag{49}$$
$$S_{23} \leq 2 S_{13} \tag{50}$$
$$S_{24} \leq 2 S_{14} \tag{51}$$
$$S_{14} + S_{24} + S_{34} \geq 120 \tag{52}$$
$$S_{14} \geq 12 \tag{53}$$
$$S_{24} \geq 25 \tag{54}$$
$$S_{34} \geq 16 \tag{55}$$
$$x_{it}, S_{it} \geq 0,\ \ i = 1, 2, 3,\ t = 1, 2, 3, 4. \tag{56}$$

Expression (23) maximizes the present profit. Constraints (24) to (26) ensure, for each species, that the amount of fish captured in the first year does not exceed the maximum amount that may exist in the fishery. Constraints (27) to (29) determine, for each species, the amount of fish that is in the fishery at the end of year 1. Constraints (30) to (47) guarantee the same for the remaining years. Constraints (48) to (51) ensure, for each year, that the amount of tuna in the fishery is less than or equal to twice the amount of salmon. Constraint (52) states that the total amount of fish in the fishery at the end of the 4-year period is greater than or equal to 120 t. Constraints (53) to (55) establish, for each species, a minimum amount of fish in the fishery at the end of the period. Constraints (56) state the non-negativity requirements on the variables.

2. **Solve the model and criticize the solution.**

   The optimal solution obtained is displayed in the following table. The present profit of this solution is 335609.54 €.

	x^*_{it} (ton)			
	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$i = 1$	0	0	0	28.19
$i = 2$	0	0	0	40.64
$i = 3$	0	0	0	0

   There is no profit in the first three years. Sardines are not selected to be captured during the 4-year period, which can threaten the ecological equilibrium.

3. **Add constraints to the model to ensure an even profit over time.**

$$\sum_{i=1}^{3} p_{it} x_{it} \leq 1.15 \sum_{i=1}^{3} p_{i,t-1} x_{i,t-1},\ t = 2, 3, 4 \tag{57}$$
$$\sum_{i=1}^{3} p_{it} x_{it} \geq 0.85 \sum_{i=1}^{3} p_{i,t-1} x_{i,t-1},\ t = 2, 3, 4 \tag{58}$$

Constraints (57) and (58) ensure that the profit obtained in each period neither is larger than 1.15 times the profit obtained in the previous period nor smaller than 0.85 times this profit.

The optimal solution obtained with these new constraints is displayed in the following table. The present profit of this solution is 273736.51 €.

Not only there is a profit every year as also sardine is captured.

	x^*_{it} (t)			
	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$i = 1$	7.85	0	0	1.50
$i = 2$	4.51	0	0	3.01
$i = 3$	0	17.20	15.21	8.33

Question 3.

To think

Consider a linear programming model with constraint C_i (model A) and the model where constraint C_i is replaced by constraint $C_j = \frac{1}{10}C_i$ (model B). Let p_i and p_j denote the shadow prices of constraints C_i and C_j, and s_i and s_j the slack values of these constraints.

1. **What is the relationship between p_i and p_j?**

   [See table below]

- Models A e B are equivalent. Let Z^* denote the optimal objective value for both models.

- Let D_j be $LHS_j \leq RHS_j + \Delta$, and Z_1^* the optimal objective value obtained by increasing the RHS of C_j by Δ.

- Let D_i be $10LHS_j \leq 10RHS_j + 10\Delta$. Thus, Z_1^* is also the optimal objective value obtained by increasing the RHS of C_i by 10Δ.

Model	Constraint	$LHS \leq RHS$	Optimal solution
B	C_j	$LHS_j \leq RHS_j$	Z^*
A	$C_i = 10C_j$	$10LHS_j \leq 10RHS_j$	Z^*
B	D_j	$LHS_j \leq RHS_j + \Delta$	Z_1^*
A	$D_i = 10D_j$	$10LHS_j \leq 10RHS_j + 10\Delta$	Z_1^*

Therefore,

- $s_j = \dfrac{Z_1^* - Z^*}{RHS_j + \Delta - RHS_j} = \dfrac{Z_1^* - Z^*}{\Delta}$ and

- $s_i = \dfrac{Z_1^* - Z^*}{10RHS_j + 10\Delta - 10RHS_j} = \dfrac{Z_1^* - Z^*}{10\Delta} = \dfrac{1}{10}\dfrac{Z_1^* - Z^*}{\Delta} = \dfrac{1}{10}s_j$.

2. **What is the relationship between s_i and s_j?**

   Let LHS^* and $10LHS^*$ be the left-hand-sides of constraints (j) and (i) with the optimal activity levels.

Model	Constraint	$LHS \leq RHS$	Slack value
B	C_j	$LHS_j^* \leq RHS_j$	$s_j = RHS_j - LHS_j^*$
A	$C_i = 10C_j$	$10LHS_j^* \leq 10RHS_j$	$s_i = 10RHS_j - 10LHS_j^* =$
			$= 10(RHS_j - LHS_j^*)$

- $s_i = 10s_j$.