# The Simplex method at a glance

March 8

Isabel Martins

ISA - Applied Operations Research - 2022/2023

# Vertices

# Keeping the river clean

A pulp mill makes mechanical and chemical pulp and during the production process it pollutes the river in which it spills its spent waters. The owners would like to minimize pollution, keeping at least 300 people employed at the mill and generating at least 40000 € of revenue per day.

# Keeping the river clean - Data

- The maximum capacity of the mill is 300 tons per day to make mechanical pulp and 200 tons per day to make chemical pulp (the mechanical pulp line cannot be used to make chemical pulp, and vice-versa)
- Both mechanical and chemical pulp require the labor of 1 worker for about 1 day, or 1 workday (wd), per ton produced
- Pollution is measured by the biological oxygen demand (BOD). 1 ton of mechanical pulp produces 1 unit of BOD, 1 ton of chemical pulp produces 1.5 units
- The chemical pulp sells at 200 €, the mechanical pulp at 100 € per ton.

# Keeping the river clean - Decision variables and formulation

$x_1$ - Amount of mechanical pulp produced (in tons/day, or t/d)

$x_2$ - Amount of chemical pulp produced (t/d).

$$\min Z = x_1 + 1.5x_2 \quad \text{units of BOD per day} \tag{1}$$

*subject to*

$$x_1 + x_2 \geq 300 \quad \text{workers employed per day} \tag{2}$$

$$100x_1 + 200x_2 \geq 40000 \quad \text{revenue, €/day} \tag{3}$$

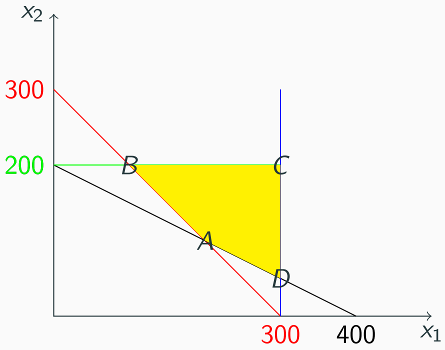
$$x_1 \leq 300 \quad \text{mechanical pulping capacity, t/day} \tag{4}$$

$$x_2 \leq 200 \quad \text{chemical pulping capacity, t/day} \tag{5}$$

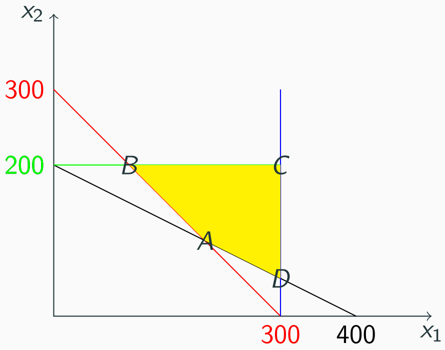
$$x_1 \geq 0 \tag{6}$$

$$x_2 \geq 0. \tag{7}$$

# Keeping the river clean - Feasible region and vertices

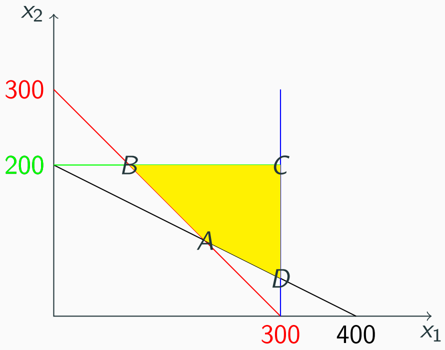


# Keeping the river clean - Feasible region and vertices



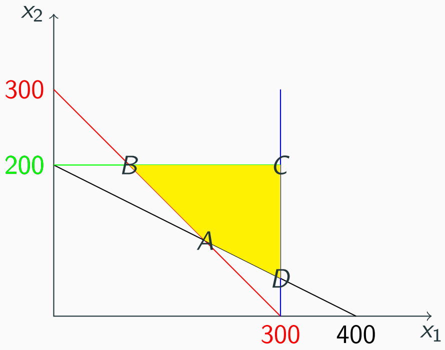

$$A = \begin{cases} x_1 + x_2 & = & 300 \\ 100x_1 + 200x_2 & = & 40000 \end{cases} \Leftrightarrow \begin{cases} x_1 & = & 200 \\ x_2 & = & 100 \end{cases} \longrightarrow Z = 350$$

# Keeping the river clean - Feasible region and vertices



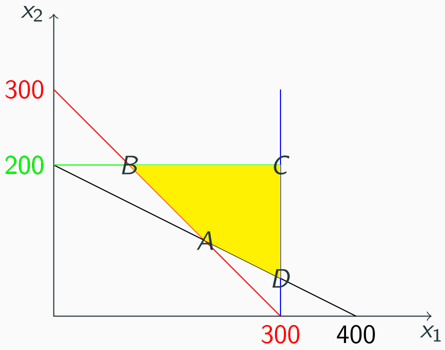

$$B = \begin{cases} x_1 + x_2 = 300 \\ x_2 = 200 \end{cases} \Leftrightarrow \begin{cases} x_1 = 100 \\ x_2 = 200 \end{cases} \longrightarrow Z = 400$$

# Keeping the river clean - Feasible region and vertices




$$C = \begin{cases} x_2 = 200 \\ x_1 = 300 \end{cases} \longrightarrow Z = 600$$

# Keeping the river clean - Feasible region and vertices




$$D = \begin{cases} x_1 & = & 300 \\ 100x_1 + 200x_2 & = & 40000 \end{cases} \Leftrightarrow \begin{cases} x_1 & = & 300 \\ x_2 & = & 50 \end{cases} \longrightarrow Z = 375$$

# Linear systems

$$\text{Linear system} \begin{cases} \text{inconsistent} & \text{Solution set empty} \\ \text{consistent dependent} & \text{Solution set with infinite solutions} \\ \text{consistent independent} & \text{Solution set with a unique solution} \end{cases}$$

Em português

$$\text{Linear system} \begin{cases} \text{“impossível”} & \text{Solution set empty} \\ \text{“possível indeterminado”} & \text{Solution set with infinite solutions} \\ \text{“possível determinado”} & \text{Solution set with a unique solution} \end{cases}$$

# Properties of vertices

For a LPP with $k$ variables.

1. A vertex is a feasible solution that is the unique solution of a system with $k$ constraints in the equality.

# Properties of vertices

For a LPP with $k$ variables.

1. A vertex is a feasible solution that is the unique solution of a system with $k$ constraints in the equality.

**Example:** Keeping the river clean - 2 variables

| System | Solution | Violated constraints |
|---|---|---|
| $\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $A(200, 100)$ | - |
| $\begin{cases} x_1 + x_2 = 300 \\ x_2 = 200 \end{cases}$ | $B(100, 200)$ | - |
| $\begin{cases} x_2 = 200 \\ x_1 = 300 \end{cases}$ | $C(300, 200)$ | - |
| $\begin{cases} x_1 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $D(300, 50)$ | - |

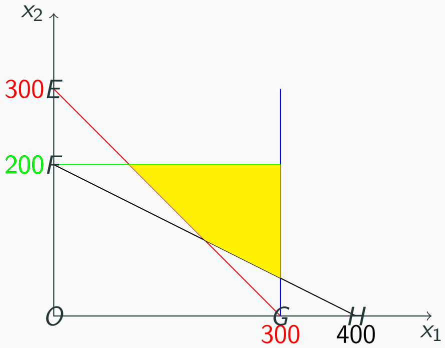
# Properties of vertices

It is possible to have systems with $k$ constraints in the equality consistent independent that do not define vertices.

# Properties of vertices

It is possible to have systems with $k$ constraints in the equality consistent independent that do not define vertices.

**Example:** Keeping the river clean - 2 variables




$$E = \begin{cases} x_1 + x_2 = 300 \\ x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 300 \end{cases}$$

# Properties of vertices

| System | Solution | Violated constraints |
|---|---|---|
| $\begin{cases} x_1 + x_2 = 300 \\ x_1 = 0 \end{cases}$ | $E(0, 300)$ | $x_2 \leq 200$ |
| $\begin{cases} x_1 = 0 \\ x_2 = 200 \end{cases}$ | $F(0, 200)$ | $x_1 + x_2 \geq 300$ |
| $\begin{cases} x_1 = 0 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $F(0, 200)$ | $x_1 + x_2 \geq 300$ |
| $\begin{cases} x_2 = 200 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $F(0, 200)$ | $x_1 + x_2 \geq 300$ |
| $\begin{cases} x_1 = 300 \\ x_2 = 0 \end{cases}$ | $G(300, 0)$ | $100x_1 + 200x_2 \geq 40000$ |
| $\begin{cases} x_1 = 300 \\ x_1 + x_2 = 300 \end{cases}$ | $G(300, 0)$ | $100x_1 + 200x_2 \geq 40000$ |
| $\begin{cases} x_1 + x_2 = 300 \\ x_2 = 0 \end{cases}$ | $G(300, 0)$ | $100x_1 + 200x_2 \geq 40000$ |

# Properties of vertices

| System | Solution | Violated constraints |
|---|---|---|
| $\begin{cases} x_2 = 0 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $H(400, 0)$ | $x_1 \leq 300$ |
| $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ | $O(0, 0)$ | $x_1 + x_2 \geq 300$<br>$100x_1 + 200x_2 \geq 40000$ |
| $\begin{cases} x_1 = 300 \\ x_1 = 0 \end{cases}$ | $\emptyset$ | - |
| $\begin{cases} x_2 = 200 \\ x_2 = 0 \end{cases}$ | $\emptyset$ | - |

$C_2^6 = 15$ systems: 4 are vertices; 9 correspond to unfeasible solutions; 2 are inconsistent.

# Properties of vertices

Can systems with $k$ constraints in the equality be consistent dependent?

# Properties of vertices

**Example:** Keeping the river clean with one more constraint

$$\min\ Z = x_1 + 1.5x_2 \tag{8}$$
$$x_1 + x_2 \geq 300 \tag{9}$$
$$100x_1 + 100x_2 \geq 30000 \tag{10}$$
$$100x_1 + 200x_2 \geq 40000 \tag{11}$$
$$x_1 \leq 300 \tag{12}$$
$$x_2 \leq 200 \tag{13}$$
$$x_1, x_2 \geq 0. \tag{14}$$

| Consistent dependent system | Solution |
|---|---|
| $\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 100x_2 = 30000 \end{cases}$ | $x_1 + x_2 = 300$ |

# Properties of vertices

Consider the LPP (with $k$ variables) with all variables $\geq 0$. In this case, if the LPP is feasible, there are vertices!

# Properties of vertices

Consider the LPP (with $k$ variables) with all variables $\geq 0$. In this case, if the LPP is feasible, there are vertices!

2. The number of vertices if finite.
   The maximum number is $C_k^{m+p}$, where $m$ is the number of constraints and $p$ is the number of signal constraints.

# Properties of vertices

Consider the LPP (with $k$ variables) with all variables $\geq 0$. In this case, if the LPP is feasible, there are vertices!

2. The number of vertices if finite.
   The maximum number is $C_k^{m+p}$, where $m$ is the number of constraints and $p$ is the number of signal constraints.
3. If there is just one optimal solution, this solution is a vertex.

# Properties of vertices

Consider the LPP (with $k$ variables) with all variables $\geq 0$. In this case, if the LPP is feasible, there are vertices!

2. The number of vertices if finite.
   The maximum number is $C_k^{m+p}$, where $m$ is the number of constraints and $p$ is the number of signal constraints.
3. If there is just one optimal solution, this solution is a vertex.

Consider the LPP with a bounded feasible region

# Properties of vertices

Consider the LPP (with $k$ variables) with all variables $\geq 0$. In this case, if the LPP is feasible, there are vertices!

2. The number of vertices if finite.
   The maximum number is $C_k^{m+p}$, where $m$ is the number of constraints and $p$ is the number of signal constraints.
3. If there is just one optimal solution, this solution is a vertex.

Consider the LPP with a bounded feasible region

4. If there are alternative optimal solutions, these solutions are vertices and any convex combination of the vertices (a linear combination of the vertices in which the coefficients are non-negative and all add up to one). For a LPP with 2 variables, these solutions are a pair of vertices and any solution that lies on the straight line segment joining both vertices.

# Properties of vertices

Consider the LPP (with $k$ variables) with all variables $\geq 0$. In this case, if the LPP is feasible, there are vertices!

2. The number of vertices if finite.
   The maximum number is $C_k^{m+p}$, where $m$ is the number of constraints and $p$ is the number of signal constraints.
3. If there is just one optimal solution, this solution is a vertex.

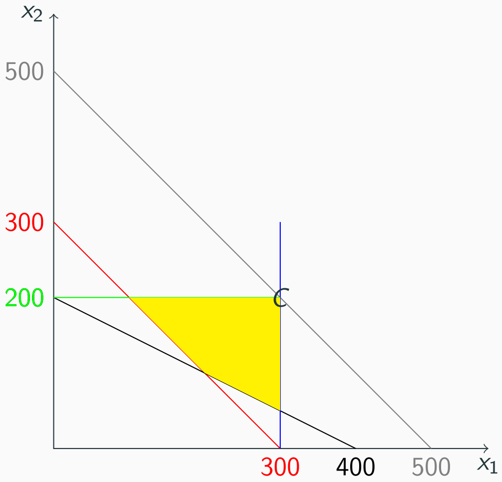
Consider the LPP with a bounded feasible region

4. If there are alternative optimal solutions, these solutions are vertices and any convex combination of the vertices (a linear combination of the vertices in which the coefficients are non-negative and all add up to one). For a LPP with 2 variables, these solutions are a pair of vertices and any solution that lies on the straight line segment joining both vertices.
5. If a vertex does not have adjacent vertices with a better objective function then there are no better feasible solutions.

# Degenerate vertex

**Definition**
A degenerate vertex is a feasible solution that is the unique solution of more than one system with $k$ constraints in the equality.

# Degenerate vertex




$$C = \begin{cases} x_1 = 300 \\ x_2 = 200 \end{cases} = \begin{cases} x_1 = 300 \\ x_1 + x_2 = 500 \end{cases} = \begin{cases} x_2 = 200 \\ x_1 + x_2 = 500 \end{cases}$$

# The core idea of the Simplex method

# The core idea of the Simplex method

Make all variables non-negative

Find an initial vertex. If there is no vertex, the problem is unfeasible and STOP

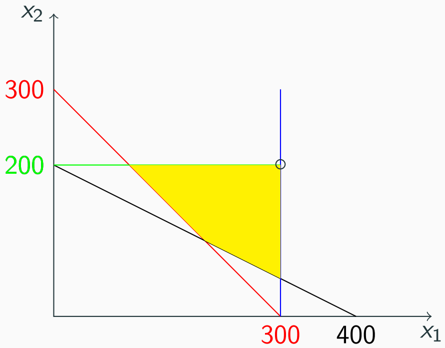
**repeat**

Verify if the objective function value can be improved. If not, an optimal solution was found and STOP

Move to the adjacent vertex in the direction that most improves the objective function. If there is no such solution, the problem is unbounded! STOP

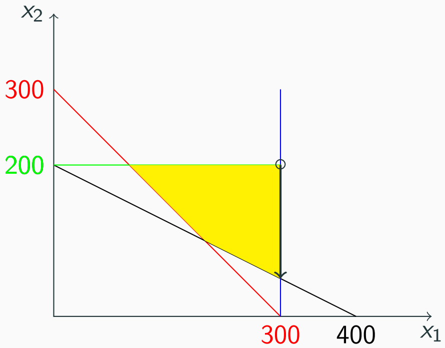
**until** Stopping criteria is fulfilled

# The core idea of the Simplex method



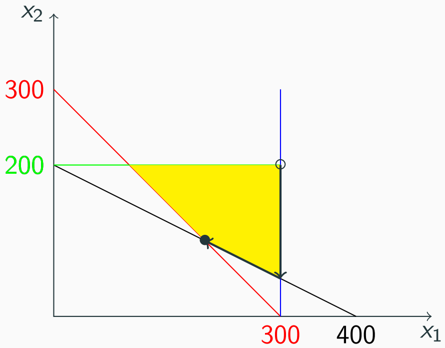

min $Z = x_1 + 1.5x_2$

# The core idea of the Simplex method




min $Z = x_1 + 1.5x_2$

# The core idea of the Simplex method




min $Z = x_1 + 1.5x_2$

# Finding the vertices

# Finding the vertices

**Definition**

LPP is in the standard form if constraints are equations and variables are $\geq 0$.

## Constraints $\geq$

**Example:**

$x_1 + x_2 \geq 300 \Leftrightarrow$

# Standard form

## Constraints $\geq$

**Example:**

$$x_1 + x_2 \geq 300 \Leftrightarrow \begin{cases} x_1 + x_2 - 300 = s_1 \\ s_1 \geq 0 \end{cases} \Leftrightarrow$$

## Constraints $\geq$

**Example:**

$$x_1 + x_2 \geq 300 \Leftrightarrow \begin{cases} x_1 + x_2 - 300 = s_1 \\ s_1 \geq 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x_1 + x_2 - s_1 = 300 \\ s_1 \geq 0 \end{cases}$$

# Standard form

## Constraints $\geq$

**Example:**

$$x_1 + x_2 \geq 300 \Leftrightarrow \begin{cases} x_1 + x_2 - 300 = s_1 \\ s_1 \geq 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x_1 + x_2 - s_1 = 300 \\ s_1 \geq 0 \end{cases}$$

$s_1$ **- Number of workers employed beyond the minimum required**

## Constraints $\geq$

**Example:**

$$x_1 + x_2 \geq 300 \Leftrightarrow \begin{cases} x_1 + x_2 - 300 = s_1 \\ s_1 \geq 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x_1 + x_2 - s_1 = 300 \\ s_1 \geq 0 \end{cases}$$

$s_1$ - **Number of workers employed beyond the minimum required**

**Generalizing**

$LHS \geq RHS \Leftrightarrow$

## Constraints $\geq$

**Example:**

$x_1 + x_2 \geq 300 \Leftrightarrow \begin{cases} x_1 + x_2 - 300 = s_1 \\ s_1 \geq 0 \end{cases} \Leftrightarrow$

$\begin{cases} x_1 + x_2 - s_1 = 300 \\ s_1 \geq 0 \end{cases}$

$s_1$ - **Number of workers employed beyond the minimum required**

**Generalizing**

$LHS \geq RHS \Leftrightarrow \begin{cases} LHS - s = RHS \\ s \geq 0 \end{cases}$

# Standard form

## Constraints $\leq$

**Example:**

$x_1 \leq 300 \Leftrightarrow$

# Standard form

## Constraints $\leq$

**Example:**

$$x_1 \leq 300 \Leftrightarrow \begin{cases} 300 - x_1 = s_3 \\ s_3 \geq 0 \end{cases} \Leftrightarrow$$

# Standard form

## Constraints $\leq$

**Example:**

$$x_1 \leq 300 \Leftrightarrow \begin{cases} 300 - x_1 = s_3 \\ s_3 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + s_3 = 300 \\ s_3 \geq 0 \end{cases}$$

# Standard form

## Constraints $\leq$

**Example:**

$$x_1 \leq 300 \Leftrightarrow \begin{cases} 300 - x_1 = s_3 \\ s_3 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + s_3 = 300 \\ s_3 \geq 0 \end{cases}$$

$s_3$ - **Mechanical pulp capacity that is not used (t/day)**

# Standard form

## Constraints $\leq$

**Example:**

$$x_1 \leq 300 \Leftrightarrow \begin{cases} 300 - x_1 = s_3 \\ s_3 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + s_3 = 300 \\ s_3 \geq 0 \end{cases}$$

$s_3$ - **Mechanical pulp capacity that is not used (t/day)**

**Generalizing**

$$LHS \leq RHS \Leftrightarrow \begin{cases} LHS + s = RHS \\ s \geq 0 \end{cases}$$

# Standard form

**Constraints** $=$

**Do nothing!**

# Standard form

**Constraints** $=$

**Do nothing!**

**Objective function**

**Do nothing!**

# Standard form

**Example:** Keeping the river clean

$x_1$ - Amount of mechanical pulp produced (in tons/day, or t/d)

$x_2$ - Amount of chemical pulp produced (t/d)

$s_1$ - Number of workers employed beyond the minimum required

$s_2$ - Revenue obtained beyond the minimum required (euros/day)

$s_3$ - Mechanical pulping capacity that is not used (t/day)

$s_4$ - Chemical pulping capacity that is not used (t/day)

# Standard form

$$\min Z = x_1 + 1.5x_2 \quad \text{units of BOD per day} \tag{15}$$

*subject to*

$$x_1 + x_2 - s_1 = 300 \quad \text{workers employed} \tag{16}$$

$$100x_1 + 200x_2 - s_2 = 40000 \quad \text{revenue, euros/day} \tag{17}$$

$$x_1 + s_3 = 300 \quad \text{mechanical pulping capacity, t/day} \tag{18}$$

$$x_2 + s_4 = 200 \quad \text{chemical pulping capacity, t/day} \tag{19}$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0. \tag{20}$$

# Vertices and basic feasible solutions

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : \left| \begin{array}{ccccc} x_1 & + & x_2 & \geq & 300 \\ 100x_1 & + & 200x_2 & \geq & 40000 \\ x_1 & & & \leq & 300 \\ & & x_2 & \leq & 200 \end{array} \right. , x_1, x_2 \geq 0\}$$

$$\mathcal{F} = \{(x_1, x_2, s_1, s_2, s_3, s_4) \in \mathbb{R}^6 :$$
$$\left| \begin{array}{cccccccccccc} x_1 & + & x_2 & - & s_1 & & & & & & = & 300 \\ 100x_1 & + & 200x_2 & & & - & s_2 & & & & = & 40000 \\ x_1 & & & & & & & + & s_3 & & = & 300 \\ & & x_2 & & & & & & & + \; s_4 & = & 200 \\ x_1, & & x_2, & & s_1, & & s_2, & & s_3, & s_4 & \geq & 0 \end{array} \right. \}$$

**Assumption for further results:** No equation (in $\mathcal{F}$) is a direct consequence of the others and thus unnecessary.

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

Example:

$$(250, 200) \in \mathcal{P} \longrightarrow$$

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

Example:

$$(250, 200) \in \mathcal{P} \longrightarrow$$
$$(250, 200, 150, 25000, 50, 0) \in \mathcal{F}$$

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

Example:

$$(250, 200) \in \mathcal{P} \longrightarrow$$
$$(250, 200, 150, 25000, 50, 0) \in \mathcal{F}$$

$$(200, 200, 100, 20000, 100, 0) \in \mathcal{F} \longrightarrow$$

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

Example:

$$(250, 200) \in \mathcal{P} \longrightarrow$$
$$(250, 200, 150, 25000, 50, 0) \in \mathcal{F}$$

$$(200, 200, 100, 20000, 100, 0) \in \mathcal{F} \longrightarrow$$
$$(200, 200) \in \mathcal{P}$$

# Vertices and basic feasible solutions

Consider a LPP with all variables $\geq 0$, $\mathcal{P}$ is the feasible region and $\mathcal{F}$ is the feasible region in the standard form where no equation is unnecessary.

Each point in $\mathcal{P}$ corresponds to a point in $\mathcal{F}$ and vice-versa.

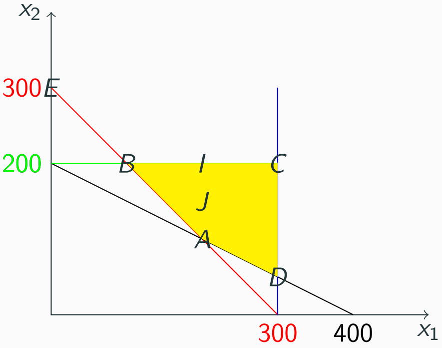
Example:

$$(250, 200) \in \mathcal{P} \longrightarrow$$
$$(250, 200, 150, 25000, 50, 0) \in \mathcal{F}$$

$$(200, 200, 100, 20000, 100, 0) \in \mathcal{F} \longrightarrow$$
$$(200, 200) \in \mathcal{P}$$

Each vertex in $\mathcal{P}$ corresponds to a ? in $\mathcal{F}$ and vice-versa.

# Vertices and basic feasible solutions



# Vertices and basic feasible solutions

| | | $(x_1, x_2)$ | $(x_1, x_2, s_1, s_2, s_3, s_4)$ |
|---|---|---|---|
| $A =$ | $\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $(200, 100)$ | $(200, 100, 0, \mathbf{0}, 100, 100)$ |
| $B =$ | $\begin{cases} x_1 + x_2 = 300 \\ x_2 = 200 \end{cases}$ | $(100, 200)$ | $(100, 200, 0, 10000, 200, 0)$ |
| $C =$ | $\begin{cases} x_2 = 200 \\ x_1 = 300 \end{cases}$ | $(300, 200)$ | $(300, 200, 200, 30000, 0, 0)$ |
| $D =$ | $\begin{cases} x_1 = 300 \\ 100x_1 + 200x_2 = 40000 \end{cases}$ | $(300, 50)$ | $(300, 50, 50, 0, \mathbf{0}, 150)$ |
| $I$ | | $(200, 200)$ | $(200, 200, 100, 20000, 100, \mathbf{0})$ |
| $J$ | | $(200, 150)$ | $(200, 150, 50, 10000, 100, 50)$ |
| $E =$ | $\begin{cases} x_1 = 0 \\ x_1 + x_2 = 300 \end{cases}$ | $(0, 300)$ | $(\mathbf{0}, 300, 0, 20000, 300, -100)$ |

# Vertices and basic feasible solutions

| | $(x_1, x_2)$ | $(x_1, x_2, s_1, s_2, s_3.s_4)$ | |
|---|---|---|---|
| $A$ | $(200, 100)$ | (200,100,0,0,100,100) | $A'$ |
| $B$ | $(100, 200)$ | (100,200,0,10000,200,0) | $B'$ |
| $C$ | $(300, 200)$ | (300,200,200,30000,0,0) | $C'$ |
| $D$ | $(300, 50)$ | (300,50,50,0,0,150) | $D'$ |
| $I$ | $(200, 200)$ | (200,200,100,20000,100,0) | $I'$ |
| $J$ | $(200, 150)$ | (200,150,50,10000,100,50) | $J'$ |
| $E$ | $(0, 300)$ | (0,300,0,20000,300,-100) | $E'$ |

# Vertices and basic feasible solutions

Each vertex in $\mathcal{P}$ corresponds to a basic feasible solution in $\mathcal{F}$ and vice-versa.

# Vertices and basic feasible solutions

Each vertex in $\mathcal{P}$ corresponds to a basic feasible solution in $\mathcal{F}$ and vice-versa.

**Definition**

A basic feasible solution (in $\mathcal{F}$, with a system with $n$ variables and $m$ equations):

# Vertices and basic feasible solutions

Each vertex in $\mathcal{P}$ corresponds to a basic feasible solution in $\mathcal{F}$ and vice-versa.

**Definition**

A basic feasible solution (in $\mathcal{F}$, with a system with $n$ variables and $m$ equations):

- has all components non-negative

# Vertices and basic feasible solutions

Each vertex in $\mathcal{P}$ corresponds to a basic feasible solution in $\mathcal{F}$ and vice-versa.

**Definition**

A basic feasible solution (in $\mathcal{F}$, with a system with $n$ variables and $m$ equations):

- has all components non-negative
- has at least $n - m$ components equal to zero

# Vertices and basic feasible solutions

Each vertex in $\mathcal{P}$ corresponds to a basic feasible solution in $\mathcal{F}$ and vice-versa.

**Definition**

A basic feasible solution (in $\mathcal{F}$, with a system with $n$ variables and $m$ equations):

- has all components non-negative
- has at least $n - m$ components equal to zero
- is the unique solution of the system of equations setting the variables associated to these components equal to zero.

## Vertices and feasible basic solutions

"is the unique solution of the system of equations setting the variables associated to these components equal to zero" is really necessary? Yes!

# Vertices and feasible basic solutions

"is the unique solution of the system of equations setting the variables associated to these components equal to zero" is really necessary? Yes!

Example:

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : \left| \begin{array}{lclcl} x_1 & + & x_2 & \geq & 300 \\ 100x_1 & + & 200x_2 & \geq & 40000 \\ x_1 & & & \leq & 300 \\ & & x_2 & \leq & 200 \\ 100x_1 & + & 100x_2 & \geq & 30000 \end{array} \right. , x_1, x_2 \geq 0\}$$

$$\mathcal{F} = \{(x_1, x_2, s_1, s_2, s_3, s_4, s_5) \in \mathbb{R}^7 : \left| \begin{array}{lllllllllllllll} x_1 & + & x_2 & - & s_1 & & & & & & & & & = & 300 \\ 100x_1 & + & 200x_2 & & & - & s_2 & & & & & & & = & 40000 \\ x_1 & & & & & & & + & s_3 & & & & & = & 300 \\ & & x_2 & & & & & & & + & s_4 & & & = & 200 \\ 100x1 & + & 100x_2 & & & & & & & & & - & s_5 & = & 30000 \\ x_1, & & x_2, & & s_1, & & s_2, & & s_3, & & s_4, & & s_5 & \geq & 0 \end{array} \right. \}$$

No equation (in $\mathcal{F}$) is a direct consequence of the others.

# Vertices and basic feasible solutions

$(150, 150)$ corresponds to $(150, 150, 0, 5000, 150, 50, 0)$ which

# Vertices and basic feasible solutions

$(150, 150)$ corresponds to $(150, 150, 0, 5000, 150, 50, 0)$ which

$\longrightarrow$ has all components non-negative

# Vertices and basic feasible solutions

$(150, 150)$ corresponds to $(150, 150, 0, 5000, 150, 50, 0)$ which

$\longrightarrow$ has all components non-negative

$\longrightarrow$ has (at least) $7 - 5 = 2$ components equal to zero

# Vertices and basic feasible solutions

$(150, 150)$ corresponds to $(150, 150, 0, 5000, 150, 50, 0)$ which

- $\longrightarrow$ has all components non-negative
- $\longrightarrow$ has (at least) $7 - 5 = 2$ components equal to zero
- $\longrightarrow$ but, it is not the unique solution of the system of equations setting the variables associated to these components equal to zero.

$$\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 - s_2 = 40000 \\ x_1 + s_3 = 300 \\ x_2 + s_4 = 200 \\ 100x1 + 100x_2 = 30000 \end{cases}$$

is consistent dependent.

# Vertices and basic feasible solutions

$(150, 150)$ corresponds to $(150, 150, 0, 5000, 150, 50, 0)$ which

- $\longrightarrow$ has all components non-negative
- $\longrightarrow$ has (at least) $7 - 5 = 2$ components equal to zero
- $\longrightarrow$ but, it is not the unique solution of the system of equations setting the variables associated to these components equal to zero.

$$\begin{cases} x_1 + x_2 = 300 \\ 100x_1 + 200x_2 - s_2 = 40000 \\ x_1 + s_3 = 300 \\ x_2 + s_4 = 200 \\ 100x1 + 100x_2 = 30000 \end{cases}$$

is consistent dependent.

$(150, 150, 0, 5000, 150, 50, 0)$ is not a feasible basic solution and $(150, 150)$ is not a vertex!

# Degenerate basic feasible solution

**Definition**

A basic feasible solution is degenerate if it has more than $n - m$ null components.

# Degenerate basic feasible solution

**Definition**
A basic feasible solution is degenerate if it has more than $n - m$ null components.

A degenerate basic feasible solution corresponds to a degenerate vertex and vice-versa.

# Simplex method at a glance

# Simplex method at a glance

Convert the linear programming model to the standard form

Find an initial feasible basic solution. If there is no feasible basic solution, the problem is unfeasible and STOP

**repeat**

Verify if the objective function value can be improved. If not, an optimal solution was found and STOP

Move to the adjacent feasible basic solution in the direction that most improves the objective function. If there is no such solution, the problem is unbounded! STOP

**until** Stopping criteria is fulfilled

# Homework

- Worksheet "Simplex method at a glance"

# Bom estudo

Source: Os Espacialistas.